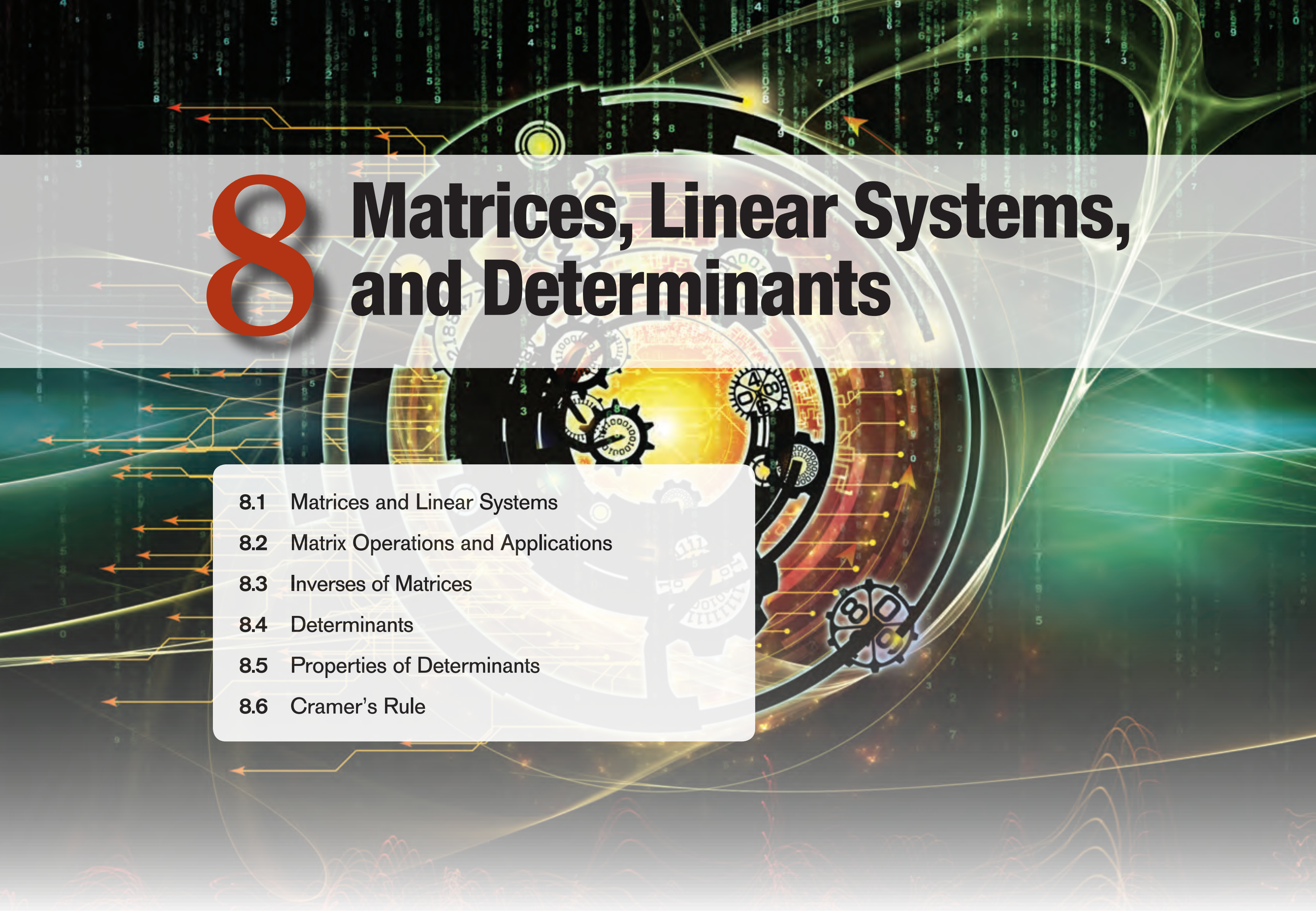


8 Matrices, Linear Systems, and Determinants

Have you recently sent or received a picture of a friend by email or cell phone? Or perhaps you plan on watching a DVD with friends tonight. The pictures you will see are digital images, which are made up of pixels. For a black-and-white image, each pixel has a value representing the gray-level intensity. If we replace each pixel in the image with its value, a number, we get a rectangular array that looks like this:

$$\begin{bmatrix} 5 & 6 & 8 \\ 4 & 0 & 1 \\ 2 & 1 & 10 \end{bmatrix}$$

This array is called a matrix. By multiplying each entry by 3, we would increase the contrast. Other matrix operations (Section 8.2) can be applied to alter the image in other ways. The chapter project explores some possibilities.

The material on matrices and determinants presented in this chapter serves as an introduction to linear algebra, a mathematical subject that is used in the natural sciences, business and economics, and the social sciences. Since methods involving matrices may require millions of numerical computations, computers have played an important role in expanding the use of matrix techniques to a wide variety of practical problems.

Our study of matrices and determinants will focus on their application to the solution of systems of linear equations. We will see that the method of Gaussian Elimination, studied in the previous chapter, can be readily implemented using matrices. We will show that matrix notation provides a convenient means for writing linear systems and that the inverse of a matrix enables us to solve such a system. Determinants will also provide us with an additional technique, known as Cramer's Rule, for the solution of certain linear systems.

It should be emphasized that this material is a very brief introduction to matrices and determinants. Their properties and applications are both extensive and important.

Explore some different topics like this and others on the Mathematical Association of America website: http://www.maa.org/press/periodicals. There you will find many uses of different concepts in mathematics.

http://www.maa.org/press/periodicals

8.1 Matrices and Linear Systems

8.1a Definitions

We have already studied several methods for solving a linear system, such as

$$\begin{aligned} 2x + 3y &= -7 \\ 3x - y &= 17 \end{aligned}$$

This system can be displayed by a **matrix**, which is a rectangular array of mn real numbers arranged in m horizontal rows and n vertical columns. The numbers are called the **entries**, or **elements**, of the matrix and are enclosed within brackets. Thus,

$$\begin{array}{cl} A = \begin{bmatrix} 2 & 3 & -7 \\ 3 & -1 & 17 \end{bmatrix} & \begin{array}{l} \leftarrow \\ \leftarrow \end{array} \text{rows} \\ \quad\;\; \uparrow \quad \uparrow \quad \uparrow & \\ \quad\;\; \text{columns} & \end{array}$$

is a matrix consisting of two rows and three columns, whose entries are obtained from the two given equations. In general, a matrix of m rows and n columns is said to be of **dimension *m* by *n***, written $m \times n$. The matrix A is seen to be of dimension 2×3. If the numbers of rows and columns of a matrix are both equal to n, the matrix is called a **square matrix of order *n***.

EXAMPLE 1 *Dimension of a Matrix*

a. $A = \begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}$

is a 2×2 matrix. Since matrix A has two rows and two columns, it is a square matrix of order 2.

b. $B = \begin{bmatrix} 4 & -5 \\ -2 & 1 \\ 3 & 0 \end{bmatrix}$

has three rows and two columns and is a 3×2 matrix.

c. $C = \begin{bmatrix} -8 & 6 & 1 \end{bmatrix}$

is a 1×3 matrix and is called a **row matrix** since it has precisely one row.

d. $D = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

is a 2×1 matrix and is called a **column matrix** since it has exactly one column. ■

8.1b *Subscript Notation*

There is a convenient way of denoting a general $m \times n$ matrix, using "double subscripts."

$$
\begin{array}{rcl}
A = & \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn}
\end{bmatrix} &
\begin{array}{l}
\leftarrow \text{first row} \\
\leftarrow \text{second row} \\
\\
\leftarrow i\text{th row} \\
\\
\leftarrow m\text{th row}
\end{array} \\
& \begin{array}{cccc}
\uparrow & \uparrow & \uparrow & \uparrow \\
\text{first column} & \text{second column} & j\text{th column} & n\text{th column}
\end{array} &
\end{array}
$$

Thus, a_{ij} is the entry in the ith row and jth column of the matrix A. It is customary to write $A = [a_{ij}]$ to indicate that a_{ij} is the entry in row i and column j of matrix A.

EXAMPLE 2 *Matrix Dimension and Element Notation*

Let

$$A = \begin{bmatrix} 3 & -2 & 4 & 5 \\ 9 & 1 & 2 & 0 \\ -3 & 2 & -4 & 8 \end{bmatrix}$$

Matrix A is of dimension 3×4. The element a_{12} is found in the first row and second column and is seen to be -2. Similarly, we see that $a_{31} = -3$, $a_{33} = -4$ and $a_{34} = 8$. ■

Progress Check

Let

$$B = \begin{bmatrix} 4 & 8 & 1 \\ 2 & -5 & 3 \\ -8 & 6 & -4 \\ 0 & 1 & -1 \end{bmatrix}$$

Find the following:

a. b_{11} b. b_{23} c. b_{31} d. b_{42}

Answers

a. 4 b. 3 c. -8 d. 1

8.1c Coefficient and Augmented Matrices

If we begin with the system of linear equations

$$\begin{aligned} 2x + 3y &= -7 \\ 3x - y &= 17 \end{aligned}$$

the matrix

$$\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$$

in which the first column is formed from the coefficients of x and the second column is formed from the coefficients of y, is called the **coefficient matrix**. The matrix

$$\left[\begin{array}{cc:c} 2 & 3 & -7 \\ 3 & -1 & 17 \end{array}\right]$$

which includes the column consisting of the right-hand sides of the equations separated by a dashed line, is called the **augmented matrix**. Note that the unknowns should always be aligned when forming the coefficient and augmented matrices.

EXAMPLE 3 *Linear Systems and the Augmented Matrix*

Write a system of linear equations that corresponds to the augmented matrix.

$$\left[\begin{array}{ccc:c} -5 & 2 & -1 & 15 \\ 0 & -2 & 1 & -7 \\ \frac{1}{2} & 1 & -1 & 3 \end{array}\right]$$

SOLUTION

We attach the unknown x to the first column, the unknown y to the second column, and the unknown z to the third column. The resulting system is

$$\begin{aligned} -5x + 2y - z &= 15 \\ -2y + z &= -7 \\ \frac{1}{2}x + y - z &= 3 \end{aligned}$$

■

Now that we have seen how a matrix can be used to represent a system of linear equations, we next proceed to show how operations on that matrix can yield the solution of the system. These "matrix methods" are simply a streamlining of the methods already studied in the previous chapter.

In Section 7.3, we used three elementary operations to transform a system of linear equations into triangular form. When applying the same procedures to a matrix, we speak of rows, columns, and elements instead of equations, unknowns, and coefficients. The three elementary operations that yield an equivalent system now become the **elementary row operations**.

Elementary Row Operations

The following elementary row operations transform an augmented matrix into another augmented matrix. These augmented matrices correspond to equivalent linear systems.

1. Interchange any two rows.
2. Multiply each element of any row by a constant $k \neq 0$.
3. Replace each element of a given row by the sum of itself plus k times the corresponding element of any other row.

The method of **Gaussian Elimination**, introduced in Section 7.3, can now be restated in terms of matrices. By use of elementary row operations we seek to transform an augmented matrix into a matrix for which $a_{ij} = 0$ when $i > j$. The resulting matrix has the following appearance for a system of three linear equations in three unknowns:

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right]$$

Since this matrix represents a linear system in triangular form, back-substitution provides a solution of the original system. We will illustrate the process with an example.

EXAMPLE 4 *Elementary Row Operations and Gaussian Elimination*

Solve the system.

$$\begin{aligned} x - y + 4z &= 4 \\ 2x + 2y - z &= 2 \\ 3x - 2y + 3z &= -3 \end{aligned}$$

SOLUTION

We describe and illustrate the steps of the procedure.

Step 1. Form the augmented matrix.	*Step 1.* The augmented matrix is $\left[\begin{array}{ccc\|c} \mathbf{1} & -1 & 4 & 4 \\ 2 & 2 & -1 & 2 \\ 3 & -2 & 3 & -3 \end{array}\right]$
Step 2. If necessary, interchange rows to make sure that a_{11}, the first element of the first row, is nonzero. We call a_{11} the **pivot element** and row 1 the **pivot row**.	*Step 2.* We see that $a_{11} = 1 \neq 0$. The pivot element is a_{11} and is shown in **blue**.
Step 3. Arrange to have 0 as the first element of every row below row 1. This is done by replacing row 2, row 3, and so on by the sum of itself and an appropriate multiple of row 1.	*Step 3.* To make $a_{21} = 0$, replace row 2 by the sum of itself and (-2) times row 1. To make $a_{31} = 0$, replace row 3 by the sum of itself and (-3) times row 1. $\left[\begin{array}{ccc\|c} 1 & -1 & 4 & 4 \\ 0 & \mathbf{4} & -9 & -6 \\ 0 & 1 & -9 & -15 \end{array}\right]$
Step 4. Repeat the process defined by *Steps 2* and *3*, allowing row 2, row 3, and so on to play the role of the first row. Thus, row 2, row 3, and so on serve as the pivot rows, with a_{22} the pivot element of row 2, a_{33}, the pivot element of row 3, and so on.	*Step 4.* Since $a_{22} = 4 \neq 0$, it serves as the next pivot element and is shown in **blue**. To make $a_{32} = 0$, replace row 3 by the sum of itself and $\left(-\frac{1}{4}\right)$ times row 2. $\left[\begin{array}{ccc\|c} 1 & -1 & 4 & 4 \\ 0 & 4 & -9 & -6 \\ 0 & 0 & -\frac{27}{4} & -\frac{27}{2} \end{array}\right]$
Step 5. The corresponding linear system is in triangular form. Solve by back-substitution.	*Step 5.* The third row of the final matrix yields $-\frac{27}{4}z = -\frac{27}{2}$, $z = 2$. Substituting $z = 2$, we obtain from the second row of the final matrix $4y - 9z = -6$, $4y - 9(2) = -6$, $y = 3$. Substituting $y = 3$ and $z = 2$, we obtain from the first row of the final matrix $x - y + 4z = 4$, $x - 3 + 4(2) = 4$, $x = -1$. The solution is $x = -1$, $y = 3$, $z = 2$.

Progress Check

Solve the linear system by matrix methods.

$$\begin{aligned} 2x + 4y - z &= 0 \\ x - 2y - 2z &= 2 \\ -5x - 8y + 3z &= -2 \end{aligned}$$

Answer

$x = 6$, $y = -2$, $z = 4$

Note that we described the process of Gaussian Elimination in a manner that applies to any augmented matrix that is $n \times (n + 1)$. Thus, Gaussian Elimination may be used on any system of n linear equations in n unknowns that has a unique solution.

It is also permissible to perform elementary row operations in ways to simplify the arithmetic. For example, you may wish to interchange rows or multiply a row by a constant to obtain a pivot element equal to 1. We will illustrate these ideas with an example.

EXAMPLE 5 *Elementary Row Operations and Gaussian Elimination*

Solve by matrix methods.

$$\begin{aligned} 2y + 3z &= 4 \\ 4x + y + 8z + 15w &= -14 \\ x - y + 2z &= 9 \\ -x - 2y - 3z - 6w &= 10 \end{aligned}$$

SOLUTION

We begin with the augmented matrix and perform a sequence of elementary row operations. The pivot element is shown in **blue**.

Write the augmented matrix. Note that $a_{11} = 0$.

$$\left[\begin{array}{rrrr|r} 0 & 2 & 3 & 0 & 4 \\ 4 & 1 & 8 & 15 & -14 \\ 1 & -1 & 2 & 0 & 9 \\ -1 & -2 & -3 & -6 & 10 \end{array}\right]$$

Interchange rows 1 and 3 so that $a_{11} = 1$.

$$\left[\begin{array}{rrrr|r} \mathbf{1} & -1 & 2 & 0 & 9 \\ 4 & 1 & 8 & 15 & -14 \\ 0 & 2 & 3 & 0 & 4 \\ -1 & -2 & -3 & -6 & 10 \end{array}\right]$$

To make $a_{21} = 0$, replace row 2 by the sum of itself and (-4) times row 1. To make $a_{41} = 0$, replace row 4 by the sum of itself and row 1.

$$\left[\begin{array}{rrrr|r} 1 & -1 & 2 & 0 & 9 \\ 0 & 5 & 0 & 15 & -50 \\ 0 & 2 & 3 & 0 & 4 \\ 0 & -3 & -1 & -6 & 19 \end{array}\right]$$

Multiply row 2 by $\frac{1}{5}$ so that $a_{22} = 1$.

$$\left[\begin{array}{rrrr:r} 1 & -1 & 2 & 0 & 9 \\ 0 & \mathbf{1} & 0 & 3 & -10 \\ 0 & 2 & 3 & 0 & 4 \\ 0 & -3 & -1 & -6 & 19 \end{array}\right]$$

To make $a_{32} = 0$, replace row 3 by the sum of itself and (-2) times row 2. To make $a_{42} = 0$, replace row 4 by the sum of itself and 3 times row 2.

$$\left[\begin{array}{rrrr:r} 1 & -1 & 2 & 0 & 9 \\ 0 & 1 & 0 & 3 & -10 \\ 0 & 0 & 3 & -6 & 24 \\ 0 & 0 & -1 & 3 & -11 \end{array}\right]$$

Interchange rows 3 and 4 so that the next pivot is $a_{33} = -1$.

$$\left[\begin{array}{rrrr:r} 1 & -1 & 2 & 0 & 9 \\ 0 & 1 & 0 & 3 & -10 \\ 0 & 0 & \mathbf{-1} & 3 & -11 \\ 0 & 0 & 3 & -6 & 24 \end{array}\right]$$

To make $a_{43} = 0$, replace row 4 by the sum of itself and 3 times row 3.

$$\left[\begin{array}{rrrr:r} 1 & -1 & 2 & 0 & 9 \\ 0 & 1 & 0 & 3 & -10 \\ 0 & 0 & -1 & 3 & -11 \\ 0 & 0 & 0 & 3 & -9 \end{array}\right]$$

The last row of the matrix indicates that

$$\begin{aligned} 3w &= -9 \\ w &= -3 \end{aligned}$$

The remaining unknowns are found by back-substitution.

Third Row of Final Matrix	Second Row of Final Matrix	First Row of Final Matrix
$-z + 3w = -11$	$y + 3w = -10$	$x - y + 2z = 9$
$-z + \mathbf{3(-3)} = \mathbf{-11}$	$y + \mathbf{3(-3)} = \mathbf{-10}$	$x - \mathbf{(-1)} + \mathbf{2(2)} = \mathbf{9}$
$z = \mathbf{2}$	$y = \mathbf{-1}$	$x = \mathbf{4}$

The solution is $x = 4$, $y = -1$, $z = 2$, $w = -3$. ■

8.1d *Gauss-Jordan Elimination*

There is an important variant of Gaussian Elimination known as **Gauss-Jordan Elimination**. The objective of this variant is to transform a linear system into a form that yields a solution without back-substitution. For a 3 × 3 system that has a unique solution, the final matrix and equivalent linear system look like this:

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right] \qquad \begin{aligned} x + 0y + 0z &= c_1 \\ 0x + y + 0z &= c_2 \\ 0x + 0y + z &= c_3 \end{aligned}$$

The solution is then seen to be $x = c_1$, $y = c_2$ and $z = c_3$.

The execution of the Gauss-Jordan Method is essentially the same as that of Gaussian Elimination with these exceptions:

1. The pivot elements are always required to be equal to 1.
2. All elements in a column other than the pivot element are forced to be 0.

These objectives are accomplished by the use of elementary row operations as illustrated in the following example.

EXAMPLE 6 *Gauss-Jordan Elimination*

Solve the linear system by the Gauss-Jordan Method.

$$\begin{aligned} x - 3y + 2z &= 12 \\ 2x + y - 4z &= -1 \\ x + 3y - 2z &= -8 \end{aligned}$$

SOLUTION

We begin with the augmented matrix. At each stage, the pivot element is shown in **blue** and is used to force all elements in that column other than the pivot element itself to be zero.

The pivot element is a_{11}.

$$\left[\begin{array}{rrr:r} \mathbf{1} & -3 & 2 & 12 \\ 2 & 1 & -4 & -1 \\ 1 & 3 & -2 & -8 \end{array}\right]$$

To make $a_{21} = 0$, replace row 2 by the sum of itself and -2 times row 1. To make $a_{31} = 0$, replace row 3 by the sum of itself and -1 times row 1.

$$\left[\begin{array}{rrr:r} 1 & -3 & 2 & 12 \\ 0 & 7 & -8 & -25 \\ 0 & 6 & -4 & -20 \end{array}\right]$$

Replace row 2 by the sum of itself and -1 times row 3 to yield the next pivot, $a_{22} = 1$.

$$\left[\begin{array}{rrr:r} 1 & -3 & 2 & 12 \\ 0 & \mathbf{1} & -4 & -5 \\ 0 & 6 & -4 & -20 \end{array}\right]$$

To make $a_{12} = 0$, replace row 1 by the sum of itself and 3 times row 2. To make $a_{32} = 0$, replace row 3 by the sum of itself and -6 times row 2.

$$\left[\begin{array}{rrr:r} 1 & 0 & -10 & -3 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & 20 & 10 \end{array}\right]$$

Multiply row 3 by $\frac{1}{20}$ so that $a_{33} = 1$.

$$\left[\begin{array}{rrr:r} 1 & 0 & -10 & -3 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & \mathbf{1} & \frac{1}{2} \end{array}\right]$$

To make $a_{13} = 0$, replace row 1 by the sum of itself and 10 times row 3. To make $a_{23} = 0$, replace row 2 by the sum of itself and 4 times row 3.

$$\left[\begin{array}{rrr:r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

We see the solution directly from the final matrix: $x = 2$, $y = -3$, and $z = \frac{1}{2}$. ■

Graphing Calculator Power User's CORNER

Reduced Row Echelon Form

Your graphing calculator can take an augmented matrix and return the reduced row echelon form required by Gauss-Jordan Elimination. As you will see in Example 7, the dashed line customarily found in an augmented matrix does not appear. Your graphing calculator is a powerful tool for solving systems of equations. However, you must be able to interpret the information it gives you. Recall the cases in which there are either infinitely many solutions, or no solution.

EXAMPLE 7 *Gauss-Jordan Elimination Using the Graphing Calculator*

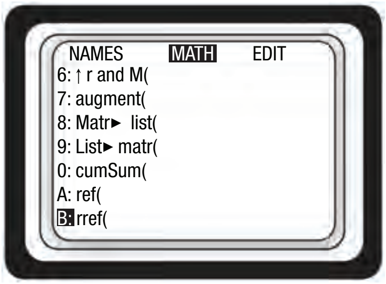
Consider the system

$$\begin{aligned} 0.03x + \quad y - 0.07z &= 0.89 \\ x - 0.01y + 0.12z &= 1.23 \\ 1.02x - 1.02y + \quad z &= 2 \end{aligned}$$

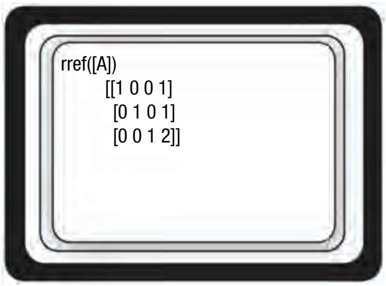
The augmented matrix for this system is

$$\begin{bmatrix} 0.03 & 1 & -0.07 & 0.89 \\ 1 & -0.01 & 0.12 & 1.23 \\ 1.02 & -1.02 & 1 & 2 \end{bmatrix}$$

After entering this matrix into the graphing calculator and naming it A, we select reduced row echelon form from the MATRIX MATH menu:



Here is the result:



The solution is (1, 1, 2). ■

Exercise Set 8.1

In Exercises 1–6, state the dimension of each matrix.

1. $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 3 & -1 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 2 & 3 \\ 5 & -1 & 4 \\ 2 & 3 & 6 \\ -8 & -1 & 2 \end{bmatrix}$

4. $\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$

5. $\begin{bmatrix} 4 & 2 & 1 \\ 3 & 1 & 5 \\ -4 & -2 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 3 & -1 & 2 & 6 \\ 2 & 8 & 4 & 1 \end{bmatrix}$

7. Given

$$A = \begin{bmatrix} 3 & -4 & -2 & 5 \\ 8 & 7 & 6 & 2 \\ 1 & 0 & 9 & -3 \end{bmatrix}$$

find

a. a_{12} b. a_{22} c. a_{23} d. a_{34}

8. Given

$$B = \begin{bmatrix} -5 & 6 & 8 \\ 4 & 1 & 3 \\ 0 & 2 & -6 \\ -3 & 9 & 7 \end{bmatrix}$$

find

a. b_{13} b. b_{21} c. b_{33} d. b_{42}

In Exercises 9–12, write the coefficient matrix and the augmented matrix for each given linear system.

9. $\begin{aligned} 3x - 2y &= 12 \\ 5x + y &= -8 \end{aligned}$

10. $\begin{aligned} 3x - 4y &= 15 \\ 4x - 3y &= 12 \end{aligned}$

11. $\begin{aligned} \tfrac{1}{2}x + y + z &= 4 \\ 2x - y - 4z &= 6 \\ 4x + 2y - 3z &= 8 \end{aligned}$

12. $\begin{aligned} 2x + 3y - 4z &= 10 \\ -3x + y \quad &= 12 \\ 5x - 2y + z &= -8 \end{aligned}$

In Exercises 13–16, write the linear system whose augmented matrix is given.

13. $\left[\begin{array}{cc|c} \frac{3}{2} & 6 & -1 \\ 4 & 5 & 3 \end{array}\right]$

14. $\left[\begin{array}{cc|c} 4 & 0 & 2 \\ -7 & 8 & 3 \end{array}\right]$

15. $\left[\begin{array}{ccc|c} 1 & 1 & 3 & -4 \\ -3 & 4 & 0 & 8 \\ 2 & 0 & 7 & 6 \end{array}\right]$

16. $\left[\begin{array}{ccc|c} 4 & 8 & 3 & 12 \\ 1 & -5 & 3 & -14 \\ 0 & 2 & 7 & 18 \end{array}\right]$

In Exercises 17–20, the augmented matrix corresponding to a linear system has been transformed to the given matrix by elementary row operations. Find a solution of the original linear system.

17. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$

18. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right]$

19. $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right]$

20. $\left[\begin{array}{ccc|c} 1 & -4 & 2 & -4 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$

In Exercises 21–30, solve the given linear system by applying Gaussian Elimination to the augmented matrix.

21. $\begin{aligned} x - 2y &= -4 \\ 2x + 3y &= 13 \end{aligned}$

22. $\begin{aligned} 2x + y &= -1 \\ 3x - y &= -7 \end{aligned}$

23. $\begin{aligned} x + y + z &= 4 \\ 2x - y + 2z &= 11 \\ x + 2y + 2z &= 6 \end{aligned}$

24. $\begin{aligned} x - y + z &= -5 \\ 3x + y + 2z &= -5 \\ 2x - y - z &= -2 \end{aligned}$

25. $\begin{aligned} 2x + y - z &= 9 \\ x - 2y + 2z &= -3 \\ 3x + 3y + 4z &= 11 \end{aligned}$

26. $\begin{aligned} 2x + y - z &= -2 \\ -2x - 2y + 3z &= 2 \\ 3x + y - z &= -4 \end{aligned}$

27. $\begin{aligned} -x - y + 2z &= 9 \\ x + 2y - 2z &= -7 \\ 2x - y + z &= -9 \end{aligned}$

28. $\begin{aligned} 4x + y - z &= -1 \\ x - y + 2z &= 3 \\ -x + 2y - z &= 0 \end{aligned}$

29. $\begin{aligned} x + y - z + 2w &= 0 \\ 2x + y \qquad - w &= -2 \\ 3x \qquad + 2z \qquad &= -3 \\ -x + 2y \qquad + 3w &= 1 \end{aligned}$

30. $\begin{aligned} 2x + y \qquad - 3w &= -7 \\ 3x \qquad + 2z + w &= 0 \\ -x + 2y \qquad + 3w &= 10 \\ -2x - 3y + 2z - w &= 7 \end{aligned}$

In Exercises 31–40, solve the linear systems of Exercises 21–30 by Gauss-Jordan Elimination applied to the augmented matrix.

In Exercises 41–50, solve the linear systems of Exercises 21–30 in your graphing calculator by using the reduced row echelon option under your MATRIX menu.

51. A black-and-white digital image has 30 rows of 18 pixels each. If the image is represented as a matrix with each entry the value of the corresponding pixel, what are the dimensions of the matrix?

52. *Mathematics in Writing:* In your own words, describe the difference between Gaussian elimination and Gauss-Jordan elimination. Which do you prefer? Why?

8.2 Matrix Operations and Applications

Now that we have defined a matrix, we can define various operations with matrices. First we begin with the definition of equality.

Equality of Matrices
Two matrices are equal if they are of the same dimension and their corresponding entries are equal.

EXAMPLE 1 *Matrix Equality*

Solve for all unknowns.

$$\begin{bmatrix} -2 & 2x & 9 \\ y-1 & 3 & -4s \end{bmatrix} = \begin{bmatrix} z & 6 & 9 \\ -4 & r & 7 \end{bmatrix}$$

SOLUTION

Equating corresponding elements, we must have

$$\begin{aligned} -2 &= z && \text{or} \quad z = -2 \\ 2x &= 6 && \text{or} \quad x = 3 \\ y - 1 &= -4 && \text{or} \quad y = -3 \\ 3 &= r && \text{or} \quad r = 3 \\ -4s &= 7 && \text{or} \quad s = -\frac{7}{4} \end{aligned}$$

■

Matrix addition can be performed only when the matrices are of the same dimension.

Matrix Addition
The sum $A + B$ of two $m \times n$ matrices A and B is the $m \times n$ matrix obtained by adding the corresponding elements of A and B.

EXAMPLE 2 *Matrix Addition*

Given the following matrices,

$$A = [2 \quad -3 \quad 4] \qquad B = [5 \quad 3 \quad 2]$$

$$C = \begin{bmatrix} 1 & 6 & -1 \\ -2 & 4 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 16 & 2 & 9 \\ 4 & -7 & -1 \end{bmatrix}$$

find (if possible):

a. $A + B$ b. $A + D$ c. $C + D$

SOLUTION

a. Since A and B are both 1×3 matrices, they can be added, giving

$$A + B = [2 + 5 \quad -3 + 3 \quad 4 + 2] = [7 \quad 0 \quad 6]$$

b. Matrices A and D are not of the same dimension and cannot be added.

c. C and D are both 2×3 matrices. Thus,

$$C + D = \begin{bmatrix} 1+16 & 6+2 & -1+9 \\ -2+4 & 4+(-7) & 5+(-1) \end{bmatrix} = \begin{bmatrix} 17 & 8 & 8 \\ 2 & -3 & 4 \end{bmatrix}$$

■

A matrix is a way of writing the information displayed in a table. For example, Table 8-1 displays the current inventory of the Quality TV Company at its various outlets.

TABLE 8-1 Inventory of Television Sets

TV Sets	Boston	Miami	Chicago
17 inch	140	84	25
19 inch	62	17	48

The same data is displayed by the matrix S, where we understand the columns to represent the cities and the rows to represent the sizes of the television sets.

$$S = \begin{bmatrix} 140 & 84 & 25 \\ 62 & 17 & 48 \end{bmatrix}$$

If the matrix

$$M = \begin{bmatrix} 30 & 46 & 15 \\ 50 & 25 & 60 \end{bmatrix}$$

specifies the number of sets of each size received at each outlet the following month, then the matrix

$$T = S + M = \begin{bmatrix} 170 & 130 & 40 \\ 112 & 42 & 108 \end{bmatrix}$$

gives the revised inventory.

Suppose the salespeople at each outlet are told that half of the revised inventory is to be placed on sale. To determine the number of sets of each size to be placed on sale, we need to multiply each element of the matrix T by 0.5. When working with matrices, we call a real number such as 0.5 a **scalar** and define **scalar multiplication** as follows.

Scalar Multiplication
To multiply a matrix A by a scalar c, multiply each entry of A by c.

EXAMPLE 3 *Scalar Multiplication*

The matrix Q

$$\begin{array}{cc} & \begin{array}{ccc} \text{Regular} & \text{Unleaded} & \text{Premium} \end{array} \\ Q = & \begin{bmatrix} 130 & 250 & 60 \\ 110 & 180 & 40 \end{bmatrix} \end{array} \begin{array}{l} \text{City A} \\ \text{City B} \end{array}$$

shows the quantity (in thousands of gallons) of the principal types of gasolines stored by a refiner at two different locations. It is decided to increase the quantity of each type of gasoline stored at each site by 10%. Use scalar multiplication to determine the desired inventory levels.

SOLUTION

To increase each entry of matrix Q by 10%, we compute the scalar product $1.1Q$.

$$1.1Q = 1.1\begin{bmatrix} 130 & 250 & 60 \\ 110 & 180 & 40 \end{bmatrix} = \begin{bmatrix} 1.1(130) & 1.1(250) & 1.1(60) \\ 1.1(110) & 1.1(180) & 1.1(40) \end{bmatrix} = \begin{bmatrix} 143 & 275 & 66 \\ 121 & 198 & 44 \end{bmatrix}$$ ■

We denote $A + (-1)B$ by $A - B$ and refer to this as the *difference* of A and B.

Matrix Subtraction
The difference $A - B$ of two $m \times n$ matrices A and B is the $m \times n$ matrix obtained by subtracting each entry of B from the corresponding entry of A.

EXAMPLE 4 *Matrix Subtraction*

Using the matrices C and D of Example 2, find $C - D$.

SOLUTION
By definition,

$$C - D = \begin{bmatrix} 1-16 & 6-2 & -1-9 \\ -2-4 & 4-(-7) & 5-(-1) \end{bmatrix} = \begin{bmatrix} -15 & 4 & -10 \\ -6 & 11 & 6 \end{bmatrix}$$

■

8.2a Matrix Multiplication

We will use the Quality TV Company again, this time to help us arrive at a definition of matrix multiplication. Suppose

$$B = \begin{array}{c} \begin{array}{ccc} \text{Boston} & \text{Miami} & \text{Chicago} \end{array} \\ \begin{bmatrix} 60 & 85 & 70 \\ 40 & 100 & 20 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{17 inch} \\ \text{19 inch} \end{array}$$

is a matrix representing the number of television sets in stock at the end of the year. Further, suppose the cost of each 17-inch set is \$80 and the cost of each 19-inch set is \$125. To find the total cost of the inventory at each outlet, we multiply the number of 17-inch sets by \$80, the number of 19-inch sets by \$125, and add the two products. If we let

$$A = [80 \quad 125]$$

be the cost matrix, we seek to define the product

$$AB = [80 \quad 125] \begin{bmatrix} 60 & 85 & 70 \\ 40 & 100 & 20 \end{bmatrix}$$

so that the result is a matrix displaying the total inventory cost at each outlet. We need to calculate for

the Boston outlet	$(80)(60) + (125)(40) = 9800$
the Miami outlet	$(80)(85) + (125)(100) = 19{,}300$
the Chicago outlet	$(80)(70) + (125)(20) = 8100$

The total inventory cost at each outlet can then be displayed by the 1×3 matrix

$$C = [9800 \quad 19{,}300 \quad 8100]$$

which is the product of A and B. Thus,

$$\begin{aligned} AB &= [80 \quad 125] \begin{bmatrix} 60 & 85 & 70 \\ 40 & 100 & 20 \end{bmatrix} \\ &= [(80)(60) + (125)(40) \quad (80)(85) + (125)(100) \quad (80)(70) + (125)(20)] \\ &= [9800 \quad 19{,}300 \quad 8100] = C \end{aligned}$$

This example illustrates the process for multiplying a 1×2 matrix times a 2×3 matrix. The general definition of matrix multiplication utilizes the same basic idea. That is, multiplication of matrices requires calculating sums of products. In this example, the first matrix had two columns and the second matrix had two rows. If we denote the elements of the first matrix as

$$A = [a_{11} \quad a_{12}]$$

and the elements of the second matrix as

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

then matrix multiplication requires that we calculate

$$[a_{11}b_{11} + a_{12}b_{21} \quad a_{11}b_{12} + a_{12}b_{22} \quad a_{11}b_{13} + a_{12}b_{23}]$$

If we denote the elements of this product by

$$C = [c_{11} \quad c_{12} \quad c_{13}]$$

then we see that

$$c_{1k} = a_{11}b_{1k} + a_{12}b_{2k} \qquad \text{for } k = 1, 2, 3$$

Matrix Multiplication

The product of AB of $m \times n$ matrix $A = [a_{ij}]$ and the $n \times r$ matrix $B = [b_{jk}]$ is obtained by forming the $m \times r$ matrix $C = [c_{ik}]$, where

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk}$$

for $i = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, r$.

It is important to note that the product AB only exists if the number of columns of A equals the number of rows of B. See Figure 8-1.

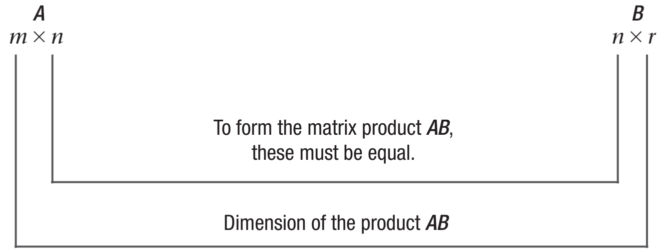


FIGURE 8-1
Dimension of the Product Matrix

EXAMPLE 5 *Matrix Multiplication*

Find the product AB if

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -6 & -2 & 4 \\ 2 & 0 & 1 & -5 \end{bmatrix}$$

SOLUTION

$$AB = \begin{bmatrix} (2)(4)+(1)(2) & (2)(-6)+(1)(0) & (2)(-2)+(1)(1) & (2)(4)+(1)(-5) \\ (3)(4)+(5)(2) & (3)(-6)+(5)(0) & (3)(-2)+(5)(1) & (3)(4)+(5)(-5) \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -12 & -3 & 3 \\ 22 & -18 & -1 & -13 \end{bmatrix}$$

■

Progress Check

Find the product AB if

$$A = \begin{bmatrix} -2 & -1 & 2 \\ 4 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & -4 \\ 3 & 1 \\ -1 & 0 \end{bmatrix}$$

Answer

$$AB = \begin{bmatrix} -15 & 7 \\ 28 & -13 \end{bmatrix}$$

EXAMPLE 6 *Matrix Multiplication*

Given the matrices

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 5 & -3 \\ -2 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 3 & -1 & -2 \\ 1 & 0 & 4 \end{bmatrix} \quad D = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

a. Show that $AB \neq BA$.

b. Determine the dimension of CD.

SOLUTION

a. $$AB = \begin{bmatrix} (1)(5)+(-1)(-2) & (1)(-3)+(-1)(2) \\ (2)(5)+(3)(-2) & (2)(-3)+(3)(2) \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} (5)(1)+(-3)(2) & (5)(-1)+(-3)(3) \\ (-2)(1)+(2)(2) & (-2)(-1)+(2)(3) \end{bmatrix} = \begin{bmatrix} -1 & -14 \\ 2 & 8 \end{bmatrix}$$

Since the corresponding elements of AB and BA are not equal, $AB \neq BA$.

b. The product of a 2×3 matrix and a 3×1 matrix is a 2×1 matrix. ■

Progress Check

If possible, find the dimension of CD and of CB, using the matrices of Example 6.

Answers

2×1; not defined

We saw in Example 6 that $AB \neq BA$; that is, the commutative law does not hold for matrix multiplication. However, the associative law $A(BC) = (AB)C$ does hold when the dimensions of A, B, and C permit us to find the necessary products.

Progress Check

Verify that $A(BC) = (AB)C$ for the matrices of A, B, and C of Example 6.

8.2b Matrices and Linear Systems

Matrix multiplication provides a convenient shorthand for writing a linear system. For example, the linear system

$$\begin{aligned} 2x - y - 2z &= 3 \\ 3x + 2y + z &= -1 \\ x + y - 3z &= 14 \end{aligned}$$

can be expressed as

$$AX = B$$

where

$$A = \begin{bmatrix} 2 & -1 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & -3 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ -1 \\ 14 \end{bmatrix}$$

To verify this, form the matrix product AX and then apply the definition of matrix equality to the matrix equation $AX = B$.

EXAMPLE 7 *Matrices and Linear Systems*

Write the linear system $AX = B$ if

$$A = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad B = \begin{bmatrix} 16 \\ -3 \end{bmatrix}$$

SOLUTION

Equating corresponding elements of the matrix equation $AX = B$ yields

$$\begin{aligned} -2x + 3y &= 16 \\ x + 4y &= -3 \end{aligned}$$

Exercise Set 8.2

1. For what values of a, b, c, and d are the matrices A and B equal?
$$A = \begin{bmatrix} a & b \\ 6 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -4 \\ c & d \end{bmatrix}$$

2. For what values of a, b, c, and d are the matrices A and B equal?
$$A = \begin{bmatrix} a+b & 2c \\ a & c-d \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 6 \\ 5 & 10 \end{bmatrix}$$

In Exercises 3–18, the following matrices are given:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -3 & 4 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 2 \\ 3 & 2 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} -3 & 2 \\ 4 & 1 \end{bmatrix}$$
$$E = \begin{bmatrix} 1 & -3 & 2 \\ 3 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$
$$G = \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

In Exercises 3–18, if possible, compute the indicated matrix.

3. $C + E$
4. $C - E$
5. $2A + 3G$
6. $3G - 4A$
7. $A + F$
8. $2B - D$
9. AB
10. BA
11. $CB + D$
12. $EB - FA$
13. $DF + AB$
14. $AC + 2DG$
15. $DA + EB$
16. $FG + B$
17. $2GE - 3A$
18. $AB + FG$

19. If
$$A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$
$$C = \begin{bmatrix} -4 & -3 \\ 0 & -4 \end{bmatrix}$$
show that $AB = AC$.

20. If
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$
show that $AB \neq BA$.

21. If
$$A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$$
show that
$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

22. If
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
show that
$$A \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

23. If
$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$
show that $AI = A$ and $IA = A$.

24. Pesticides are sprayed on plants to eliminate harmful insects. However, some of the pesticide is absorbed by the plant, and the pesticide is then absorbed by herbivores (plant-eating animals, such as cows) when they eat the plants that have been sprayed. Suppose that we have three pesticides and four plants, and that the amounts of pesticide absorbed by the different plants are given by the matrix

$$\begin{array}{cc} & \begin{array}{cccc} \text{Plant 1} & \text{Plant 2} & \text{Plant 3} & \text{Plant 4} \end{array} \\ A = & \begin{bmatrix} 3 & 2 & 4 & 3 \\ 6 & 5 & 2 & 4 \\ 4 & 3 & 1 & 5 \end{bmatrix} \end{array} \begin{array}{l} \text{Pesticide 1} \\ \text{Pesticide 2} \\ \text{Pesticide 3} \end{array}$$

where a_{ij} denotes the amount of pesticide i in milligrams that has been absorbed by plant j. Thus, plant 4 has absorbed 5 milligrams of pesticide 3. Now suppose that we have three herbivores and that the numbers of plants eaten by these animals are given by the matrix

$$\begin{array}{cc} & \begin{array}{ccc} \text{Herbivore 1} & \text{Herbivore 2} & \text{Herbivore 3} \end{array} \\ B = & \begin{bmatrix} 18 & 30 & 20 \\ 12 & 15 & 10 \\ 16 & 12 & 8 \\ 6 & 4 & 12 \end{bmatrix} \end{array} \begin{array}{l} \text{Plant 1} \\ \text{Plant 2} \\ \text{Plant 3} \\ \text{Plant 4} \end{array}$$

How much of pesticide 2 has been absorbed by herbivore 3?

25. What does entry a_{23} in the matrix product AB of Exercise 24 represent?

In Exercises 26–29, find the matrices A, X, and B so that the matrix equation $AX = B$ is equivalent to the given linear system.

26. $\begin{aligned} 7x - 2y &= 6 \\ -2x + 3y &= -2 \end{aligned}$

27. $\begin{aligned} 3x + 4y &= -3 \\ 3x - y &= 5 \end{aligned}$

28. $\begin{aligned} 5x + 2y - 3z &= 4 \\ 2x - \tfrac{1}{2}y + z &= 10 \\ x + y - 5z &= -3 \end{aligned}$

29. $\begin{aligned} 3x - y + 4z &= 5 \\ 2x + 2y + \tfrac{3}{4}z &= -1 \\ x - \tfrac{1}{4}y + z &= \tfrac{1}{2} \end{aligned}$

In Exercises 30–33, write the linear system that is represented by the matrix equation $AX = B$.

30. $A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad B = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$

31. $A = \begin{bmatrix} 1 & -5 \\ 4 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

32. $A = \begin{bmatrix} 1 & 7 & -2 \\ 3 & 6 & 1 \\ -4 & 2 & 0 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}$

33. $A = \begin{bmatrix} 4 & 5 & -2 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}$

34. The $m \times n$ matrix all of whose elements are zero is called the **zero matrix** and is denoted by 0. Show that $A + 0 = A$ for every $m \times n$ matrix A.

35. The square matrix of order n, such that $a_{ij} = 1$ and $a_{ij} = 0$ when $i \neq j$, is called the **identity matrix** of order n and is denoted by I_n. (*Note*: The definition indicates that the diagonal elements are all equal to 1 and all elements of the diagonal are 0.) Show that $AI_n = I_nA$ for every square matrix A of order n.

36. The matrix B, each of whose entries is the negative of the corresponding entry of matrix A, is called the **additive inverse** of the matrix A. Show that $A + B = 0$ where 0 is the zero matrix. (See Exercise 34.)

37. A square black-and-white digital image with 9 pixels may be represented as a matrix, like matrix A in Exercise 23. Suppose the image has 4 bits per pixel. Each bit has a value of 0 or 1. Each entry in A must be an integer between 0 (darkest black) to $2^4 - 1$, or 15 (whitest white). (*Note*: The integers from 0 to 15 represent 16 possible values.)

    Suppose $A = \begin{bmatrix} 0 & 4 & 6 \\ 5 & 0 & 1 \\ 7 & 2 & 3 \end{bmatrix}$

    The *contrast* is increased by multiplying each entry by a scaling factor. Find the matrix $2A$, representing an image with increased contrast.

38. The *digital negative image* of an image is found by subtracting each element of the image matrix from its maximum possible value. The i,j entry of the matrix N for the digital negative of A in Exercise 37 is

$$n_{ij} = 15 - a_{ij}$$

    Find the matrix N.

39. We can add one image to another and represent the resulting image by the matrix sum of the image matrix for each. Find the matrix for the image that results from adding the image represented by A to its negative N. Describe the image qualitatively. What would it look like?

8.3 Inverses of Matrices

If $a \neq 0$, then the linear equation $ax = b$ can be solved by multiplying both sides by the reciprocal of a. Thus, we obtain $x = \left(\frac{1}{a}\right)b$. It would be nice if we could multiply both sides of the matrix equation $AX = B$ by the "reciprocal of A." Unfortunately, a matrix has *no* reciprocal. However, we shall discuss a notion that, for a square matrix, provides an analogue of the reciprocal of a real number and will enable us to solve the linear system in a manner distinct from the Gauss-Jordan Method discussed earlier in this chapter.

In this section we confine our attention to square matrices. The $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

that has 1 for each entry on the main diagonal and 0 elsewhere is called the **identity matrix**. Examples of identity matrices follow:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If A is any $n \times n$ matrix, we can show that

$$AI_n = I_nA = A$$

(See Exercise 35, Section 8.2.) Thus, I_n is the matrix analogue of the real number 1.

An $n \times n$ matrix A is called **invertible**, or **nonsingular**, if we can find an $n \times n$ matrix B such that

$$AB = BA = I_n$$

The matrix B is called an **inverse** of A.

EXAMPLE 1 *Verifying Inverses*

Show that A and B are inverses of one another where

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

SOLUTION

Since

$$AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we conclude that A is an invertible matrix and that B is an inverse of A. (Verify the above equation.) Note that if B is an inverse of A, then A is an inverse of B. ■

It can be shown that if an $n \times n$ matrix A has an inverse, it can have only one inverse. We denote the inverse of A by A^{-1}. Thus, we have

$$AA^{-1} = I_n \quad \text{and} \quad A^{-1}A = I_n$$

Note that the products AA^{-1} and $A^{-1}A$ yield the *identity matrix* I_n, whereas the products $aa^{-1} = a\left(\frac{1}{a}\right)$ and $a^{-1}a = \left(\frac{1}{a}\right)a$ yield the *identity element* 1 for any real number $a \neq 0$. For this reason, A^{-1} may be thought of as the *matrix analogue* of the reciprocal $\frac{1}{a}$.

Progress Check

Verify that the matrices

$$A = \begin{bmatrix} 4 & 5 \\ 5 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & \frac{5}{2} \\ 1 & -2 \end{bmatrix}$$

are inverses of each other.

WARNING

If $a \neq 0$ is a real number, then a^{-1} has the property that $aa^{-1} = a^{-1}a = 1$. Since $a^{-1} = \frac{1}{a}$, we may refer to a^{-1} as the inverse, *or* reciprocal, of a. Although the matrix A^{-1} is the inverse of the $n \times n$ matrix A, since $AA^{-1} = A^{-1}A = I_n$, it cannot be referred to as the reciprocal of A, since *matrix division is not defined*.

We now develop a practical method for finding the inverse of an invertible matrix. Suppose we want to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Let the inverse be denoted by

$$B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$

Then we must have

$$AB = I_2 \tag{1}$$

and

$$BA = I_2 \tag{2}$$

Equation (1) now becomes

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} b_1 + 3b_3 & b_2 + 3b_4 \\ 2b_1 + 5b_3 & 2b_2 + 5b_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since two matrices are equal if, and only if, their corresponding entries are equal, we have

$$\begin{aligned} b_1 + 3b_3 &= 1 \\ 2b_1 + 5b_3 &= 0 \end{aligned} \tag{3}$$

and

$$\begin{aligned} b_2 + 3b_4 &= 0 \\ 2b_2 + 5b_4 &= 1 \end{aligned} \tag{4}$$

We solve the linear systems (3) and (4) by Gauss-Jordan Elimination. We begin with the augmented matrices of the linear systems and perform a sequence of elementary row operations as follows:

	(3)	(4)
Write the augmented matrices of (3) and (4).	$\left[\begin{array}{cc\|c} 1 & 3 & 1 \\ 2 & 5 & 0 \end{array}\right]$	$\left[\begin{array}{cc\|c} 1 & 3 & 0 \\ 2 & 5 & 1 \end{array}\right]$
To make $a_{21} = 0$, replace row 2 with the sum of itself and -2 times row 1.	$\left[\begin{array}{cc\|c} 1 & 3 & 1 \\ 0 & -1 & -2 \end{array}\right]$	$\left[\begin{array}{cc\|c} 1 & 3 & 0 \\ 0 & -1 & 1 \end{array}\right]$
Multiply row 2 by -1 to obtain $a_{22} = 1$.	$\left[\begin{array}{cc\|c} 1 & 3 & 1 \\ 0 & 1 & 2 \end{array}\right]$	$\left[\begin{array}{cc\|c} 1 & 3 & 0 \\ 0 & 1 & -1 \end{array}\right]$
To make $a_{12} = 0$, replace row 1 with the sum of itself and -3 times row 2.	$\left[\begin{array}{cc\|c} 1 & 0 & -5 \\ 0 & 1 & 2 \end{array}\right]$	$\left[\begin{array}{cc\|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$

Thus, $b_1 = -5$ and $b_3 = 2$ is the solution of (3), and $b_2 = 3$ and $b_4 = -1$ is the solution of (4). Check that

$$B = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

also satisfies the requirement $BA = I_2$ of Equation (2).

Observe that the linear systems (3) and (4) have the same coefficient matrix (which is also the same as the original matrix A) and that an identical sequence of elementary row operations was performed in the Gauss-Jordan Elimination. This suggests that we can solve the systems at *the same time*. We write the coefficient matrix A and next to it list the right-hand sides of (3) and (4) to obtain the matrix

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \tag{5}$$

Note that the columns to the right of the dashed line in (5) form the identity matrix I_2. Performing the same sequence of elementary row operations on matrix (5) as we did on matrices (3) and (4) yields

$$\left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right] \tag{6}$$

Then A^{-1} is the matrix to the right of the dashed line in (6).

The procedure outlined for the 2×2 matrix A applies in general. Thus, we have the following method for finding the inverse of an invertible $n \times n$ matrix A.

Computing A^{-1}

Step 1. Form the $n \times 2n$ matrix $[A \mid I_n]$ by adjoining the identity matrix I_n to the given matrix A.

Step 2. Apply elementary row operations to the matrix $[A \mid I_n]$ to transform the matrix A to I_n.

Step 3. The final matrix is of the form $[I_n \mid B]$ where B is A^{-1}.

EXAMPLE 2 *Computing Inverses*

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 1 & 1 & 1 \end{bmatrix}$$

SOLUTION

We form the 3×6 matrix $[A \mid I_3]$ and transform it by elementary row operations to the form $[I_3 \mid A^{-1}]$. The pivot element at each stage is shown in **blue**.

Write matrix A augmented by I_3.

$$\left[\begin{array}{ccc|ccc} \mathbf{1} & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 7 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

To make $a_{21} = 0$, replace row 2 with the sum of itself and -2 times row 1. To make $a_{31} = 0$, replace row 3 by the sum of itself and -1 times row 1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & \mathbf{1} & 1 & -2 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right]$$

To make $a_{12} = 0$, replace row 1 with the sum of itself and -2 times row 2. To make $a_{32} = 0$, replace row 3 with the sum of itself and row 2.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -3 & 1 & 1 \end{array}\right]$$

Multiply row 3 by -1.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 3 & -1 & -1 \end{array}\right]$$

To make $a_{13} = 0$, replace row 1 with the sum of itself and -1 times row 3. To make $a_{23} = 0$, replace row 2 with the sum of itself and -1 times row 3.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & -5 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{array}\right]$$

The final matrix is of the form $[I_3 \mid A^{-1}]$ that is,

$$A^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ -5 & 2 & 1 \\ 3 & -1 & -1 \end{bmatrix}$$

■

We now have a practical method for finding the inverse of an invertible matrix, but we do not know whether a given square matrix *has* an inverse. It can be shown that if the preceding procedure is carried out with the matrix $[A \mid I_n]$ and we arrive at a point at which all possible candidates for the next pivot element are zero, then the matrix is not invertible; and we may stop our calculations.

EXAMPLE 3 *Computing Inverses*

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 0 & 2 \\ -3 & -6 & -9 \end{bmatrix}$$

SOLUTION

We begin with $[A \mid I_3]$.

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ -3 & -6 & -9 & 0 & 0 & 1 \end{array}\right]$$

To make $a_{31} = 0$, replace row 3 by the sum of itself and 3 times row 1.

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 9 & 3 & 0 & 1 \end{array}\right]$$

Note that $a_{22} = a_{32} = 0$ in the last matrix. We cannot perform any elementary row operations upon rows 2 and 3 that will produce a nonzero pivot element for a_{22}. We conclude that the matrix A does not have an inverse. ■

Progress Check

Show that the matrix A is not invertible.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 2 & 1 \\ 5 & 6 & -5 \end{bmatrix}$$

8.3a *Solving Linear Systems*

Consider a linear system of n equations in n unknowns.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{7}$$

As has already been pointed out in Section 8.2 of this chapter, we can write the linear system (7) in matrix form as

$$AX = B \tag{8}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Suppose now that the coefficient matrix A is invertible so that we can compute A^{-1}. Multiplying both sides of (8) by A^{-1}, we have

$$\begin{aligned} A^{-1}(AX) &= A^{-1}B && \\ (A^{-1}A)X &= A^{-1}B && \text{Associative law} \\ I_nX &= A^{-1}B && A^{-1}A = I_n \\ X &= A^{-1}B && I_nX = X \end{aligned}$$

Thus, we have the following result:

> If $AX = B$ is a linear system of n equations in n unknowns and if the coefficient matrix A is invertible, then the system has exactly one solution, given by
>
> $$X = A^{-1}B$$

WARNING

Since matrix multiplication is not commutative, be careful to write the solution to the system $AX = B$ as $X = A^{-1}B$ and *not* $X = BA^{-1}$.

EXAMPLE 4 *Solving a System of Linear Equations Using Inverses*

Solve the linear system by finding the inverse of the coefficient matrix.

$$\begin{aligned} x + 2y + 3z &= -3 \\ 2x + 5y + 7z &= 4 \\ x + y + z &= 5 \end{aligned}$$

SOLUTION

The coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 1 & 1 & 1 \end{bmatrix}$$

is the matrix whose inverse was obtained in Example 2 as

$$A^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ -5 & 2 & 1 \\ 3 & -1 & -1 \end{bmatrix}$$

Since

$$B = \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix}$$

we obtain the solution of the given system as

$$X = A^{-1}B = \begin{bmatrix} 2 & -1 & 1 \\ -5 & 2 & 1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -5 \\ 28 \\ -18 \end{bmatrix}$$

Thus, $x = -5$, $y = 28$, $z = -18$. ■

Progress Check

Solve the linear system by finding the inverse of the coefficient matrix.

$$\begin{aligned} x - 2y + z &= 1 \\ x + 3y + 2z &= 2 \\ -x + z &= -11 \end{aligned}$$

Answer

$x = 7$, $y = 1$, $z = -4$

The inverse of the coefficient matrix is especially useful when we need to solve a number of linear systems

$$AX = B_1,\ AX = B_2,\ \ldots,\ AX = B_k$$

where the coefficient matrix is the same, and the right-hand side changes.

EXAMPLE 5 *Solving a System of Linear Equations Using Inverses*

A steel producer makes two types of steel, regular and special. A ton of regular steel requires 2 hours in the open-hearth furnace and 5 hours in the soaking pit; a ton of special steel requires 2 hours in the open-hearth furnace and 3 hours in the soaking pit. How many tons of each type of steel can be manufactured daily if

a. the open-hearth furnace is available 8 hours per day and the soaking pit is available 15 hours per day?

b. the open-hearth furnace is available 9 hours per day and the soaking pit is available 15 hours per day?

SOLUTION

Let

x = the number of tons of regular steel to be made
y = the number of tons of special steel to be made

Then the total amount of time required in the open-hearth furnace is

$$2x + 2y$$

Similarly, the total amount of time required in the soaking pit is

$$5x + 3y$$

If we let b_1 and b_2 denote the number of hours that the open-hearth furnace and the soaking pit are available per day, respectively, then we have

$$\begin{aligned} 2x + 2y &= b_1 \\ 5x + 3y &= b_2 \end{aligned}$$

or

$$\begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Verify that the inverse of the coefficient matrix is

$$\begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{5}{4} & -\frac{1}{2} \end{bmatrix}$$

a. If $b_1 = 8$ and $b_2 = 15$, then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{5}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 8 \\ 15 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{5}{2} \end{bmatrix}$$

That is, $\frac{3}{2}$ tons of regular steel and $\frac{5}{2}$ tons of special steel can be manufactured daily.

b. If $b_1 = 9$ and $b_2 = 15$, then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{5}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 9 \\ 15 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{15}{4} \end{bmatrix}$$

That is, $\frac{3}{4}$ tons of regular steel and $\frac{15}{4}$ tons of special steel can be manufactured daily. ■

FOCUS on Coded Messages

Cryptography is the study of methods for encoding and decoding messages. One of the very simplest techniques for doing this involves the use of the inverse of a matrix.

A	B	C	D	E	F	G
↕	↕	↕	↕	↕	↕	↕
1	2	3	4	5	6	7
H	I	J	K	L	M	N
↕	↕	↕	↕	↕	↕	↕
8	9	10	11	12	13	14
O	P	Q	R	S	T	U
↕	↕	↕	↕	↕	↕	↕
15	16	17	18	19	20	21
V	W	X	Y	Z		
↕	↕	↕	↕	↕		
22	23	24	25	26		

Suppose that Leslie and Ronnie are drug enforcement agents in the New York City police department and that Leslie has infiltrated a major drug operation. To avoid detection, the agents communicate with each other by using coded messages. First, they agree to attach a different number to every letter of the alphabet. For example, they let A be 1, B be 2, and so on, as shown in the accompanying table. Suppose that on Thursday, Ronnie wants to send Leslie the message

STRIKE MONDAY

to indicate that the police will raid the drug operation on the following Monday. Substituting for each letter, Ronnie sends the message

$$19, 20, 18, 9, 11, 5, 13, 15, 14, 4, 1, 25 \tag{1}$$

Unfortunately, this simple code can be easily cracked by analyzing the frequency of letters in the English alphabet. A much better method involves the use of matrices.

First, Ronnie breaks the message (1) into four 3×1 matrices

$$X_1 = \begin{bmatrix} 19 \\ 20 \\ 18 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 9 \\ 11 \\ 5 \end{bmatrix} \qquad X_3 = \begin{bmatrix} 13 \\ 15 \\ 14 \end{bmatrix} \qquad X_4 = \begin{bmatrix} 4 \\ 1 \\ 25 \end{bmatrix}$$

Sometime ago, Ronnie and Leslie had jointly selected an invertible 3×3 matrix such as

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

which no one else knows. Ronnie now forms the 3×1 matrices

$$AX_1 = \begin{bmatrix} 75 \\ 57 \\ 37 \end{bmatrix} \qquad AX_2 = \begin{bmatrix} 30 \\ 25 \\ 14 \end{bmatrix} \qquad AX_3 = \begin{bmatrix} 56 \\ 42 \\ 27 \end{bmatrix} \qquad AX_4 = \begin{bmatrix} 55 \\ 30 \\ 29 \end{bmatrix}$$

and sends the message

$$75, 57, 37, 30, 25, 14, 56, 42, 27, 55, 30, 29 \tag{2}$$

To decode the message, Leslie uses the inverse of matrix A,

$$A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

and forms

$$A^{-1}\begin{bmatrix} 75 \\ 57 \\ 37 \end{bmatrix} = X_1 \qquad A^{-1}\begin{bmatrix} 30 \\ 25 \\ 14 \end{bmatrix} = X_2 \qquad A^{-1}\begin{bmatrix} 56 \\ 42 \\ 27 \end{bmatrix} = X_3 \qquad A^{-1}\begin{bmatrix} 55 \\ 30 \\ 29 \end{bmatrix} = X_4$$

which, of course, is the original message (1) and which can be understood by using the accompanying table.

If Leslie responds with the message

$$33, 21, 16, 52, 39, 14, 66, 47, 28, 52, 38, 23$$

what is Ronnie being told?

Exercise Set 8.3

In Exercises 1–4, determine whether the matrix B is the inverse of the matrix A.

1. $A = \begin{bmatrix} 2 & \frac{1}{2} \\ -1 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

2. $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 3 & 2 & -6 \\ 1 & 1 & -2 \\ -2 & -2 & 5 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 3 \\ -4 & 1 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 2 & -2 \\ -2 & -4 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

In Exercises 5–10, find the inverse of the given matrix.

5. $\begin{bmatrix} -1 & 5 \\ 2 & -4 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$

7. $\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & -4 \\ 0 & 5 & -4 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

In Exercises 11–18, find the inverse, if possible.

11. $\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$

12. $\begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 1 & 3 \\ 2 & -8 & -4 \\ -1 & 2 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 8 & 7 & -1 \\ -5 & -5 & 1 \\ -4 & -4 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

16. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 4 & 0 \\ 2 & 0 & -6 & 1 \end{bmatrix}$

In Exercises 19–24, solve the given linear system by finding the inverse of the coefficient matrix.

19. $\begin{aligned} 2x + y &= 5 \\ x - 3y &= 6 \end{aligned}$

20. $\begin{aligned} 2x - 3y &= -5 \\ 3x + y &= -13 \end{aligned}$

21. $\begin{aligned} 3x + y - z &= 2 \\ x - 2y \quad &= 8 \\ 3y + z &= -8 \end{aligned}$

22. $\begin{aligned} 3x + y - z &= 10 \\ 2x - y + z &= -1 \\ -x + y - 2z &= 5 \end{aligned}$

23. $\begin{aligned} 2x - y + 3z &= -11 \\ 3x - y + z &= -5 \\ x + y + z &= -1 \end{aligned}$

24. $\begin{aligned} 2x + 3y - 2z &= 13 \\ 4x + 2y + z &= 3 \\ y - z &= 5 \end{aligned}$

In Exercises 25–34, solve the linear systems in Exercises 21–30, Exercise Set 8.1, by finding the inverse of the coefficient matrix.

35. Solve the linear systems $AX = B_1$ and $AX = B_2$, given

$$A^{-1} = \begin{bmatrix} 3 & -2 & 4 \\ 2 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \quad B_2 = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$$

36. Solve the linear systems $AX = B_1$ and $AX = B_2$, given

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ -1 & -1 & 3 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \quad B_2 = \begin{bmatrix} 4 \\ -3 \\ -5 \end{bmatrix}$$

37. Show that the matrix

$$\begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ d & e & f \end{bmatrix}$$

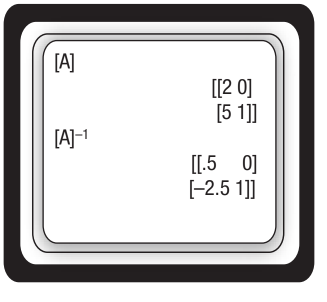
is not invertible.

38. A trustee decides to invest \$500,000 in two mortgages, which yield 4% and 8% per year, respectively. How should the \$500,000 be invested in the two mortgages if the total annual interest is to be

a. \$30,000? b. \$40,000? c. \$50,000?

(*Hint*: Some of these investment objectives cannot be attained.)

39. Many graphing calculators can find the inverse of a matrix, just by entering the name of the matrix you have stored and then hitting the inverse key. The display looks like this:



Use this method to find the inverse of the coefficient matrix for the system of equations in Example 5 in Section 8.1.

40. Now use the inverse you found in Exercise 39 to solve the system, verifying the solution given at the end of the example.
41. *Mathematics in Writing*: Explain in your own words how the inverse of a matrix is used to solve a system of equations. How is this process similar to the method for solving a linear equation in one unknown, discussed in Section 2.1? How is it different?

8.4 Determinants

In this section, we will define a determinant and develop manipulative skills for evaluating determinants. We will then show that determinants have important applications and can be used to solve linear systems.

Associated with every square matrix A is a number called the **determinant** of A, denoted by $|A|$. If A is 1×1, that is, if $A = [a_{11}]$, then we define $|A| = a_{11}$. If A is the 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then $|A|$ is said to be a *determinant of second order* and is defined by the rule

$$|A| = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

EXAMPLE 1 *Determinant of Second Order*

Compute the real number represented by

$$\begin{vmatrix} 4 & -5 \\ 3 & -1 \end{vmatrix}$$

SOLUTION

We apply the rule for a determinant of second order.

$$\begin{vmatrix} 4 & -5 \\ 3 & -1 \end{vmatrix} = (4)(-1) - (3)(-5) = 11$$

■

Progress Check

Compute the real number represented by

a. $\begin{vmatrix} -6 & 2 \\ -1 & -2 \end{vmatrix}$ b. $\begin{vmatrix} \frac{1}{2} & \frac{1}{4} \\ -4 & -2 \end{vmatrix}$

Answers

a. 14 b. 0

8.4a Minors and Cofactors

Consider the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The **minor** of an element a_{ij} is the determinant of the matrix remaining after deleting the row and column in which the element a_{ij} appears. Given the matrix

$$\begin{bmatrix} 4 & 0 & -2 \\ 1 & -6 & 7 \\ -3 & 2 & 5 \end{bmatrix}$$

the minor of the element in row 2, column 3 is

$$\begin{vmatrix} 4 & 0 & -2 \\ 1 & -6 & 7 \\ -3 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ -3 & 2 \end{vmatrix} = 8 - 0 = 8$$

The **cofactor** of the element a_{ij} is the minor of the element a_{ij} multiplied by $(-1)^{i+j}$. Since $(-1)^{i+j}$ is $+1$ if $i+j$ is even and -1 if $i+j$ is odd, we see that the cofactor is the minor with a sign attached. The cofactor attaches the sign to the minor according to this pattern:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

EXAMPLE 2 *Determining Cofactors*

Find the cofactor of each element in the first row of the matrix.

$$\begin{bmatrix} -2 & 0 & 12 \\ -4 & 5 & 3 \\ 7 & 8 & -6 \end{bmatrix}$$

SOLUTION

The cofactors are

$$(-1)^{1+1} \begin{vmatrix} -2 & 0 & 12 \\ -4 & 5 & 3 \\ 7 & 8 & -6 \end{vmatrix} = \begin{vmatrix} 5 & 3 \\ 8 & -6 \end{vmatrix} = -30 - 24 = -54$$

$$(-1)^{1+2} \begin{vmatrix} -2 & 0 & 12 \\ -4 & 5 & 3 \\ 7 & 8 & -6 \end{vmatrix} = -\begin{vmatrix} -4 & 3 \\ 7 & -6 \end{vmatrix} = -(24 - 21) = -3$$

$$(-1)^{1+3} \begin{vmatrix} -2 & 0 & 12 \\ -4 & 5 & 3 \\ 7 & 8 & -6 \end{vmatrix} = \begin{vmatrix} -4 & 5 \\ 7 & 8 \end{vmatrix} = -32 - 35 = -67$$ ■

Progress Check

Find the cofactor of each entry in the second column of the matrix.

$$\begin{bmatrix} 16 & -9 & 3 \\ -5 & 2 & 0 \\ -3 & 4 & -1 \end{bmatrix}$$

Answers

cofactor of -9 is -5; cofactor of 2 is -7; cofactor of 4 is -15

The cofactor is the key to the process of evaluating determinants of any order.

Expansion by Cofactors

To evaluate a determinant, form the sum of the products obtained by multiplying each entry of any row or any column by its cofactor. This process is called **expansion by cofactors**.

Consider the matrix

$$A = \begin{bmatrix} 4 & -5 \\ 3 & -1 \end{bmatrix}$$

and choose the second column. The cofactor of

$$a_{12} = -5 \qquad \text{is} \qquad (-1)^{1+2}\begin{bmatrix} 4 & -5 \\ 3 & -1 \end{bmatrix} = -3$$

and the cofactor of

$$a_{22} = -1 \qquad \text{is} \qquad (-1)^{2+2}\begin{bmatrix} 4 & -5 \\ 3 & -1 \end{bmatrix} = 4$$

Therefore

$$|A| = (-5)(-3) + (-1)(4) = 15 - 4 = 11$$

Note that the above is an alternative method for Example 1. In fact, verify the formula given for a determinant of order 2 at the beginning of this section using the method of expansion by cofactors, using any row or any column.

Let us illustrate the process for a 3×3 matrix.

EXAMPLE 3 *Expansion by Cofactors*

Evaluate the determinant of the matrix

$$\begin{bmatrix} -2 & 7 & 2 \\ 6 & -6 & 0 \\ 4 & 10 & -3 \end{bmatrix}$$

using the method of expansion by cofactors.

SOLUTION

Step 1. Choose a row or column about which to expand. (In general, a row or column containing zeros simplifies the work.	*Step 1.* We expand about column 3.
Step 2. Expand about the cofactors of the chosen row or column by multiplying each entry of the row or column by its cofactor. Repeat the procedure until all determinants are of order 2.	*Step 2.* The expansion about column 3 is $(2)(-1)^{1+3}\begin{vmatrix} 6 & -6 \\ 4 & 10 \end{vmatrix}$ $+(0)(-1)^{2+3}\begin{vmatrix} -2 & 7 \\ 4 & 10 \end{vmatrix}$ $+(-3)(-1)^{3+3}\begin{vmatrix} -2 & 7 \\ 6 & -6 \end{vmatrix}$
Step 3. Evaluate the cofactors and form the sum indicated in *Step 2.*	*Step 3.* Using the rule for evaluating a determinant of order 2, we have $(2)(1)[(6)(10)-(4)(-6)]+0+(-3)(1)[(-2)(-6)-(6)(7)]$ $= 2(60+24)-3(12-42)$ $= 258$

Observe that it was unnecessary for us to calculate the cofactor corresponding to the 0 element in column 3. We only did it here to reinforce the method of finding cofactors. ■

Note that expansion by cofactors of *any row or any column* produces the same result. This property of determinants can be used to simplify the arithmetic. The best choice of a row or column about which to expand is generally the one that has the most zero entries. If an entry is zero, the entry times its cofactor is also zero, so we do not have to evaluate that cofactor.

Progress Check

Find the determinant of the matrix in Example 3 by expanding about the second row.

Answer

258

EXAMPLE 4 *Expansion by Cofactors*

Verify the rule for evaluating the determinant of the matrix of order 3.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$$

SOLUTION

Expanding about the first row, we have

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \end{aligned}$$

(Verify this answer using any other column or row.) ■

Progress Check

Show that the determinant of the matrix is equal to zero.

$$\begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix}$$

The process of expanding by cofactors works for determinants of any order. If we apply the method to a determinant of order 4, we produce determinants of order 3; applying the method again results in determinants of order 2.

EXAMPLE 5 *Expansion by Cofactors*

Evaluate the determinant of the matrix.

$$\begin{vmatrix} -3 & 5 & 0 & -1 \\ 1 & 2 & 3 & -3 \\ 0 & 4 & -6 & 0 \\ 0 & -2 & 1 & 2 \end{vmatrix}$$

SOLUTION

Expanding about the cofactors of the first column, we have

$$\begin{aligned} \begin{vmatrix} -3 & 5 & 0 & -1 \\ 1 & 2 & 3 & -3 \\ 0 & 4 & -6 & 0 \\ 0 & -2 & 1 & 2 \end{vmatrix} &= -3\begin{vmatrix} 2 & 3 & -3 \\ 4 & -6 & 0 \\ -2 & 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 5 & 0 & -1 \\ 4 & -6 & 0 \\ -2 & 1 & 2 \end{vmatrix} \\ &= -3\left[-4\begin{vmatrix} 3 & -3 \\ 1 & 2 \end{vmatrix} - 6\begin{vmatrix} 2 & -3 \\ -2 & 2 \end{vmatrix}\right] - 1\left[-1\begin{vmatrix} 4 & -6 \\ -2 & 1 \end{vmatrix} + 2\begin{vmatrix} 5 & 0 \\ 4 & -6 \end{vmatrix}\right] \\ &= -3[(-4)(9) - 6(-2)] - 1[(-1)(-8) + 2(-30)] \\ &= -3[-36 + 12] - 1[8 - 60] \\ &= -3(-24) - 1(-52) = 124 \end{aligned}$$

■

Progress Check

Evaluate the determinant of the matrix.

$$\begin{vmatrix} 0 & -1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & -3 \\ 1 & 0 & 1 & 0 \end{vmatrix}$$

Answer

7

Exercise Set 8.4

In Exercises 1–6, evaluate the determinant of the given matrix.

1. $\begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix}$

2. $\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix}$

3. $\begin{vmatrix} -4 & 1 \\ 0 & 2 \end{vmatrix}$

4. $\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix}$

5. $\begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix}$

6. $\begin{vmatrix} -4 & -1 \\ -2 & 3 \end{vmatrix}$

In Exercises 7–10, let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & -3 \\ 5 & -2 & -0 \end{bmatrix}$$

7. Compute the minor of each of the following elements:
   a. a_{11} b. a_{23} c. a_{31} d. a_{33}
8. Compute the minor of each of the following elements:
   a. a_{12} b. a_{22} c. a_{23} d. a_{32}
9. Compute the cofactor of each of the following elements:
   a. a_{11} b. a_{23} c. a_{31} d. a_{33}
10. Compute the cofactor of each of the following elements:
    a. a_{12} b. a_{22} c. a_{23} d. a_{32}

In Exercises 11–20, evaluate the determinant of the given matrix.

11. $\begin{vmatrix} 4 & -2 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{vmatrix}$

12. $\begin{vmatrix} 4 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -4 \end{vmatrix}$

13. $\begin{vmatrix} -1 & 2 & 0 \\ 3 & 4 & 1 \\ 6 & 5 & 2 \end{vmatrix}$

14. $\begin{vmatrix} -1 & 3 & 2 \\ 0 & 7 & 7 \\ 2 & 1 & 3 \end{vmatrix}$

15. $\begin{vmatrix} 0 & -1 & 0 & 3 \\ 0 & 1 & 2 & 1 \\ 2 & -2 & 2 & 3 \\ 3 & 3 & 1 & 0 \end{vmatrix}$

16. $\begin{vmatrix} 0 & 0 & 2 & 3 \\ -1 & 1 & -2 & 3 \\ 0 & 2 & 2 & 1 \\ 3 & 1 & 3 & 0 \end{vmatrix}$

17. $\begin{vmatrix} 2 & 1 & -3 & 1 \\ 2 & 0 & 3 & -5 \\ 1 & -1 & 2 & 2 \\ 0 & -1 & 1 & 3 \end{vmatrix}$

18. $\begin{vmatrix} 2 & 2 & 1 & 0 \\ 1 & 0 & -1 & -1 \\ -3 & 3 & 2 & 1 \\ 1 & -5 & 2 & 3 \end{vmatrix}$

19. $\begin{vmatrix} 0 & 0 & 2 & 4 \\ 0 & 1 & 2 & 1 \\ 5 & 1 & 3 & 3 \\ 3 & 3 & 1 & 0 \end{vmatrix}$

20. $\begin{vmatrix} 3 & 2 & 0 & -1 \\ -2 & 3 & 1 & 0 \\ -2 & -2 & 4 & 4 \\ 1 & -5 & 2 & 3 \end{vmatrix}$

21. Finding the determinants by using your graphing calculator's MATRIX menu, investigate what happens to the determinant of the matrix in Exercise 17 if you change the matrix in the following ways:

    a. Interchange row 2 and row 3.

    b. Interchange row 2 and row 3, then the new row 3 and row 4.

    What can you conclude from these results?

8.5 Properties of Determinants

In general, the computations required to evaluate the determinant of a matrix can get rather time-consuming as the dimension of the matrix becomes quite large. Therefore, it may be worthwhile to consider alternative methods that may reduce the number of operations involved. We have already observed that if an element of a matrix equals zero, then we need not evaluate the corresponding cofactor since the product of the two is also zero. Thus, we will examine methods to enable us to obtain more zero entries in a matrix whose determinant is equal to that of the original matrix under consideration.

In Section 8.1, we presented the elementary row operations:

1. Interchange any two rows.
2. Multiply each element of any row by a constant $k \neq 0$.
3. Replace each element of a given row by the sum of itself plus k times the corresponding element of any other row.

We have observed that these operations are important in transforming one matrix into another matrix. We wish to explore what effect these operations have on the determinant of the original matrix compared with the determinant of the transformed matrix. We also wish to examine the determinant of some special matrices:

1. a matrix with a row of zeros
2. a matrix where two rows are identical
3. a matrix where the rows and columns are interchanged, called the **transpose** of the original matrix

Let

$$A = \begin{bmatrix} 5 & 0 & -1 \\ 4 & -6 & 0 \\ -2 & 1 & 2 \end{bmatrix}$$

Expanding about the cofactors of the first row, we find that

$$|A| = -52$$

Interchanging rows 1 and 2 of A, we have

$$\begin{vmatrix} 4 & -6 & 0 \\ 5 & 0 & -1 \\ -2 & 1 & 2 \end{vmatrix} = 52$$

Multiplying the second row of A by $\frac{1}{2}$, we obtain

$$\begin{vmatrix} 5 & 0 & -1 \\ 2 & -3 & 0 \\ -2 & 1 & 2 \end{vmatrix} = -26$$

Adding 2 times row 1 to row 3, we find that

$$\begin{vmatrix} 5 & 0 & -1 \\ 4 & -6 & 0 \\ 8 & 1 & 0 \end{vmatrix} = -52$$

If we replace the second row of A with 0 elements, we have

$$\begin{vmatrix} 5 & 0 & -1 \\ 0 & 0 & 0 \\ -2 & 1 & 2 \end{vmatrix} = 0$$

If we replace row 3 of A with row 2, we obtain

$$\begin{vmatrix} 5 & 0 & -1 \\ 4 & -6 & 0 \\ 4 & -6 & 0 \end{vmatrix} = 0$$

Taking the transpose of A, where we interchange the rows and columns of A, or equivalently, replacing a_{ij} with a_{ji} we find that

$$\begin{vmatrix} 5 & 4 & -2 \\ 0 & -6 & 1 \\ -1 & 0 & 2 \end{vmatrix} = -52$$

(Verify the calculations of the previous determinants, expanding by any row or any column.) These examples suggest the properties of determinants shown in Table 8-2.

Note that the determinant of a matrix expanded by cofactors yields the same answer, whether the expansion uses a particular row or a particular column. This fact allows us to replace the word "row" by the word "column" and obtain the same property. If a row or column has all zero entries, then expansion by cofactors about this zero row or column produces a determinant of 0. If two rows or two columns are identical, then we may add -1 times one to the other to produce a row or column of zeros, respectively.

TABLE 8-2 Properties of Determinants

1. Interchange any two rows of A or interchange any two columns of A, and call the new matrix B. Then $$\lvert B\rvert = -\lvert A\rvert$$
2. Multiply each element of any row of A or any column of A by a constant k, and call the new matrix B. Then $$\lvert B\rvert = k\lvert A\rvert$$
3. Add k times one row to any other row or k times one column to any other column and call the new matrix B. Then $$\lvert B\rvert = \lvert A\rvert$$
4. If A has a row or column with 0 elements or if A has two identical rows or two identical columns then $$\lvert A\rvert = 0$$
5. Take the transpose of A, where we replace a_{ij} with a_{ji}, so that the rows become columns and the columns become rows. If we call the new matrix B, then $$\lvert B\rvert = \lvert A\rvert$$

EXAMPLE 1 *Using Properties of Determinants*

Evaluate the determinant.

$$\begin{vmatrix} 0 & 2 & 3 & 0 \\ 4 & 1 & 8 & 15 \\ 1 & -1 & 2 & 0 \\ -1 & -2 & -3 & -6 \end{vmatrix}$$

SOLUTION

To make $a_{21} = 0$, replace row 2 by the sum of itself and (-4) times row 3. To make $a_{41} = 0$, replace row 4 by the sum of itself and row 3.

$$\begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 5 & 0 & 15 \\ 1 & -1 & 2 & 0 \\ 0 & -3 & -1 & -6 \end{vmatrix}$$

Now expand the determinant by the cofactors of the first column, obtaining

$$\begin{vmatrix} 2 & 3 & 0 \\ 5 & 0 & 15 \\ -3 & -1 & -6 \end{vmatrix}$$

We factor out 5 from the second row to obtain

$$5\begin{vmatrix} 2 & 3 & 0 \\ 1 & 0 & 3 \\ -3 & -1 & -6 \end{vmatrix}$$

To make $a_{23} = 0$, replace column 3 by the sum of itself and (-3) time column 1.

$$5\begin{vmatrix} 2 & 3 & -6 \\ 1 & 0 & 0 \\ -3 & -1 & 3 \end{vmatrix}$$

Expand this determinant by the cofactors of the second row, obtaining

$$-5\begin{vmatrix} 3 & -6 \\ -1 & 3 \end{vmatrix}$$

Evaluating this last 2×2 determinant, we have $-5(9-6) = -15$

■

Progress Check

Evaluate the determinant.

$$\begin{vmatrix} 4 & 0 & 0 & 3 \\ 2 & 4 & 5 & 8 \\ -2 & 1 & 0 & 2 \\ -4 & -1 & -2 & -3 \end{vmatrix}$$

Answer

-10

Exercise Set 8.5

In Exercises 1–6, evaluate the determinant of the given matrix.

1. $\begin{vmatrix} 2 & 2 & 4 \\ 3 & 8 & 1 \\ 1 & 1 & 2 \end{vmatrix}$

2. $\begin{vmatrix} 0 & 1 & 3 \\ 2 & 5 & -1 \\ 4 & 2 & -2 \end{vmatrix}$

3. $\begin{vmatrix} 3 & 2 & 1 & 0 \\ -1 & -3 & -1 & 0 \\ 0 & 0 & 2 & 2 \\ 4 & 1 & 3 & 3 \end{vmatrix}$

4. $\begin{vmatrix} -1 & 2 & 4 & 0 \\ 3 & -2 & -3 & 0 \\ 0 & 4 & 2 & 5 \\ 0 & -3 & 1 & 4 \end{vmatrix}$

5. $\begin{vmatrix} 2 & -3 & 2 & -4 \\ 0 & 4 & -1 & 9 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & -1 \end{vmatrix}$

6. $\begin{vmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 4 & -1 \\ -2 & 3 & 1 & -4 \\ 0 & 2 & 0 & 2 \end{vmatrix}$

7. Show that

$$\begin{vmatrix} a_1+b_1 & a_2+b_2 \\ c & d \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ c & d \end{vmatrix} + \begin{vmatrix} b_1 & b_2 \\ c & d \end{vmatrix}$$

8. Prove that if a row or column of a square matrix consists entirely of zeros, the determinant of the matrix is zero. (*Hint*: Expand by cofactors.)

9. Prove that if matrix B is obtained by multiplying each element of a row of a square matrix A by a constant k, then $|B| = k|A|$.

10. Show that

$$\begin{vmatrix} ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ ka_{21} & ka_{22} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

11. Prove that if A is an $n \times n$ matrix and $B = kA$, where k is a constant, then $|B| = k^n|A|$.

12. Prove that if matrix B is obtained from a square matrix A by interchanging the rows and columns of A, then $|B| = |A|$.

8.6 Cramer's Rule

Determinants provide a convenient way of expressing formulas in many areas of mathematics, particularly in geometry. One of the better known uses of determinants is for solving systems of linear equations, a procedure known as *Cramer's Rule*.

In an earlier section, we solved systems of linear equations by the method of elimination. We now apply this method to the general system of two equations in two unknowns.

$$a_{11}x + a_{12}y = b_1 \tag{1}$$

$$a_{21}x + a_{22}y = b_2 \tag{2}$$

Let us multiply Equation (1) by a_{22}, Equation (2) by $-a_{12}$ and add. This eliminates y.

$$\begin{array}{r} a_{11}a_{22}x + a_{12}a_{22}y = b_1a_{22} \\ -a_{21}a_{12}x - a_{12}a_{22}y = -b_2a_{12} \\ \hline a_{11}a_{22}x - a_{21}a_{12}x = b_1a_{22} - b_2a_{12} \end{array}$$

Thus,

$$x(a_{11}a_{22} - a_{21}a_{12}) = b_1a_{22} - b_2a_{12}$$

or

$$x = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}}$$

Similarly, multiplying Equation (1) by a_{21}, Equation (2) by $-a_{11}$ and adding, we can eliminate x and solve for y.

$$y = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$$

The denominators in the expression for x and y are identical and can be written as the determinant of the matrix

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

If we apply this same idea to the numerators, we have

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{|A|}, \quad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{|A|}, \quad |A| \neq 0$$

This formula is called **Cramer's Rule** and is a means of expressing the solution of a system of linear equations in determinant form. Let A_1 denote the matrix obtained by replacing the first column of A with the column of the right-hand sides of the equations. Furthermore, let A_2 denote the matrix obtained by replacing the second column of A again with the column of the right-hand sides.

We may summarize Cramer's Rule as follows:

Cramer's Rule for Two Unknowns

The solution to

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

is given by

$$x = \frac{|A_1|}{|A|}, \quad y = \frac{|A_2|}{|A|}, \quad |A| \neq 0$$

The following example outlines the steps for using Cramer's Rule.

EXAMPLE 1 *Cramer's Rule*

Solve by Cramer's Rule.

$$\begin{aligned} 3x - y &= 9 \\ x + 2y &= -4 \end{aligned}$$

SOLUTION

Step 1. Compute $\|A\|$, the determinant of the coefficient matrix A. If $A = 0$, Cramer's Rule cannot be used. Use Gaussian Elimination or Gauss-Jordan Elimination.	*Step 1.* $\|A\| = \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 7$
Step 2. Compute $\|A_1\|$, the determinant of the matrix obtained from A by replacing the column of coefficients of x, the first column unknown, with the column of right-hand sides of the equations. $x = \dfrac{\|A_1\|}{\|A\|}$	*Step 2.* $x = \dfrac{\|A_1\|}{\|A\|} = \dfrac{\begin{vmatrix} 9 & -1 \\ -4 & 2 \end{vmatrix}}{\|A\|}$ $= \dfrac{18-4}{7} = \dfrac{14}{7} = 2$
Step 3. Compute $\|A_2\|$, the determinant of the matrix obtained from A by replacing the column of coefficients of y, the second column unknown, with the column of right-hand sides of the equations. $y = \dfrac{\|A_2\|}{\|A\|}$	*Step 3.* $y = \dfrac{\|A_2\|}{\|A\|} = \dfrac{\begin{vmatrix} 3 & 9 \\ 1 & -4 \end{vmatrix}}{\|A\|}$ $= \dfrac{-12-9}{7} = \dfrac{-21}{7} = -3$ Thus, $x = 2$, $y = -3$

Progress Check

Solve by Cramer's Rule.

$$\begin{aligned} 2x + 3y &= -4 \\ 3x + 4y &= -7 \end{aligned}$$

Answers

$x = -5$, $y = 2$

The steps outlined in Example 1 can be applied to solve *any* system of linear equations in which the number of equations is the same as the number of unknowns and in which $|A| \neq 0$. For example, assume A is 3×3. If A_3 is the matrix obtained by replacing the third column of A with the column of right-hand sides, then we have

Cramer's Rule for Three Unknowns

The solution to

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

is given by

$$x = \frac{|A_1|}{|A|}, \quad y = \frac{|A_2|}{|A|}, \quad z = \frac{|A_3|}{|A|}, \quad |A| \neq 0$$

EXAMPLE 2 *Cramer's Rule*

Solve by Cramer's Rule.

$$\begin{aligned} 3x + 2z &= -2 \\ 2x - y &= 0 \\ 2y + 6z &= -1 \end{aligned}$$

SOLUTION

We compute the determinant of the matrix of coefficients.

$$|A| = \begin{vmatrix} 3 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & 6 \end{vmatrix} = -10$$

Then

$$x = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} -2 & 0 & 2 \\ 0 & -1 & 0 \\ -1 & 2 & 6 \end{vmatrix}}{|A|} = \frac{10}{-10} = -1$$

$$y = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 3 & -2 & 2 \\ 2 & 0 & 0 \\ 0 & -1 & 6 \end{vmatrix}}{|A|} = \frac{20}{-10} = -2$$

$$z = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 3 & 0 & -2 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{vmatrix}}{|A|} = \frac{-5}{-10} = \frac{1}{2}$$

Progress Check

Solve by Cramer's Rule.

$$\begin{aligned} 3x \qquad\quad - z &= 1 \\ -6x + 2y \qquad\quad &= -5 \\ -4y + 3z &= 5 \end{aligned}$$

Answers

$x = \frac{2}{3}$, $y = -\frac{1}{2}$, $z = 1$

WARNING

a. Each equation of the linear system must be written in the form

$$ax + by + cz = k$$

before using Cramer's Rule.

b. If $|A| = 0$, Cramer's Rule cannot be used.

Exercise Set 8.6

In Exercises 1–8, solve the given linear system by using Cramer's Rule.

1. $$\begin{aligned} 2x + y + z &= -1 \\ 2x - y + 2z &= 2 \\ x + 2y + z &= -4 \end{aligned}$$

2. $$\begin{aligned} x - y + z &= -5 \\ 3x + y + 2z &= -5 \\ 2x - y - z &= -2 \end{aligned}$$

3. $$\begin{aligned} 2x + y - z &= 9 \\ x - 2y + 2z &= -3 \\ 3x + 3y + 4z &= 11 \end{aligned}$$

4. $$\begin{aligned} 2x + y - z &= -2 \\ -2x - 2y + 3z &= 2 \\ 3x + y - z &= -4 \end{aligned}$$

5. $$\begin{aligned} -x - y + 2z &= 7 \\ x + 2y - 2z &= -7 \\ 2x - y + z &= -4 \end{aligned}$$

6. $$\begin{aligned} 4x + y - z &= -1 \\ x - y + 2z &= 3 \\ -x + 2y - z &= 0 \end{aligned}$$

7. $$\begin{aligned} x + y - z + 2w &= 0 \\ 2x + y \phantom{{}-z} - w &= -2 \\ 3x \phantom{{}+y} + 2z &= -3 \\ -x + 2y \phantom{{}-z} + 3w &= 1 \end{aligned}$$

8. $$\begin{aligned} 2x + y \phantom{{}+2z} - 3w &= -7 \\ 3x \phantom{{}+y} + 2z + w &= -1 \\ -x + 2y \phantom{{}+2z} + 3w &= 0 \\ -2x - 3y + 2z - w &= 8 \end{aligned}$$

9. *Mathematics in Writing*: Give a step-by-step method for solving systems of equations by Cramer's Rule with your graphing calculator.

10. Redo Exercises 7 and 8 using the method you outlined in Exercise 9.

Chapter Summary

Key Terms, Concepts, and Symbols

Key Ideas for Review

Topic	Page	Key Idea
Matrices	432	A matrix is a rectangular array of numbers.
Addition and Subtraction	441 443	The sum and difference of two matrices A and B can be formed only if A and B are of the same dimension.
Multiplication	443	The product AB can be formed only if the number of columns of A is the same as the number of rows of B.
Systems of Linear Equations and Matrix Notation	446	A linear system can be written in the form $AX = B$, where A is the coefficient matrix, X is a column matrix of the unknowns and B is the column matrix of the right-hand sides. The elementary row operations are an abstraction of those operations that produce equivalent systems of equations.
Gaussian and Gauss-Jordan Elimination	437	Gaussian Elimination and Gauss-Jordan Elimination both involve the use of elementary row operations on the augmented matrix corresponding to a linear system. In the case of a system of three equations with three unknowns and a unique solution, the final matrices are of this form: $\left[\begin{array}{ccc\|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right]$ Gaussian Elimination $\left[\begin{array}{ccc\|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right]$ Gauss-Jordan Elimination. If Gaussian Elimination is used, back-substitution is then performed with the final matrix to obtain the solution. If Gauss-Jordan Elimination is used, the solution can be read from the final matrix.
Inverse of a Matrix	449	The $n \times n$ matrix B is said to be the inverse of the $n \times n$ matrix A if $AB = I_n$ and $BA = I_n$. We denote the inverse of A by A^{-1}. The inverse can be computed by using elementary row operations to transform the matrix $[A \mid I_n]$ to the form $[I_n \mid B]$, in which case $B = A^{-1}$.
Solving Linear Systems	453	If the linear system $AX = B$ has a unique solution, then $X = A^{-1}B$.

continues

Topic	Page	Key Idea
Determinants	460	Associated with every square matrix is a number called a determinant. The determinant of the 1×1 matrix $A = [a]$ is $\|A\| = a$. The rule for evaluating a determinant of order 2 is $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
Evaluation by Cofactors	461	For determinants of order greater than 2, the method of expansion by cofactors may be used to reduce the problem to that of evaluating determinants of order 2. When expanding by cofactors, choosing the row or column that contains the most zeros usually simplifies the arithmetic.
Properties	467	Some useful properties of determinants follow: 1. Interchange any two rows of A or interchange any two columns of A, and call the new matrix B. Then $$\|B\| = -\|A\|$$ 2. Multiply each element of any row of A or any column of A by a constant k, and call the new matrix B. Then $$\|B\| = k\|A\|$$ 3. Add k times one row to any other row or k times one column to any other column and call the new matrix B. Then $$\|B\| = \|A\|$$ 4. If A has a row or column with 0 elements or if A has two identical rows or two identical columns then $$\|A\| = 0$$ 5. Take the transpose of A, where we replace a_{ij} with a_{ji}, so that the rows become columns and the columns become rows. If we call the new matrix B, then $$\|B\| = \|A\|$$
Cramer's Rule	471	Cramer's Rule provides a means for solving a linear system by expressing the value of each unknown as a quotient of determinants.

Review Exercises

Exercises 1–4 refer to the matrix

$$A = \begin{bmatrix} -1 & 4 & 2 & 0 & 8 \\ 2 & 0 & -3 & -1 & 5 \\ 4 & -6 & 9 & 1 & -2 \end{bmatrix}$$

1. Determine the dimension of the matrix A.
2. Find a_{24}.
3. Find a_{31}.
4. Find a_{15}.

Exercises 5 and 6 refer to the linear system.

$$\begin{aligned} 3x - 7y &= 14 \\ x + 4y &= 6 \end{aligned}$$

5. Write the coefficient matrix of the linear system.
6. Write the augmented matrix of the linear system.

In Exercises 7 and 8, write a linear system corresponding to the augmented matrix.

7. $\left[\begin{array}{cc:c} 4 & -1 & 3 \\ 2 & 5 & 0 \end{array}\right]$

8. $\left[\begin{array}{ccc:c} -2 & 4 & 5 & 0 \\ 6 & -9 & 4 & 0 \\ 3 & 2 & -1 & 0 \end{array}\right]$

In Exercises 9–12, use back-substitution to solve the linear system corresponding to the given augmented matrix.

9. $\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -4 \end{array}\right]$

10. $\left[\begin{array}{cc|c} 1 & 2 & \frac{21}{2} \\ 0 & 1 & 5 \end{array}\right]$

11. $\left[\begin{array}{ccc|c} 1 & -4 & 2 & -18 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right]$

12. $\left[\begin{array}{ccc|c} 1 & -2 & 2 & -9 \\ 0 & 1 & 3 & -8 \\ 0 & 0 & 1 & -3 \end{array}\right]$

In Exercises 13–16, use matrix methods to solve the given linear system.

13. $\begin{aligned} x + y &= 2 \\ 2x - 4y &= -5 \end{aligned}$

14. $\begin{aligned} 3x - y &= -17 \\ 2x + 3y &= -4 \end{aligned}$

15. $\begin{aligned} x + 3y + 2z &= 0 \\ -2x \qquad + 3z &= -12 \\ 2x - 6y - z &= 6 \end{aligned}$

16. $\begin{aligned} 2x - y - 2z &= 3 \\ -2x + 3y + z &= 3 \\ 2y - z &= 6 \end{aligned}$

In Exercises 17 and 18, solve for x.

17. $\begin{bmatrix} 5 & -1 \\ 3 & 2x \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 3 & -6 \end{bmatrix}$

18. $\begin{bmatrix} 6 & x^2 \\ 4x & -2 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ -12 & -2 \end{bmatrix}$

Exercises 19–28 refer to the following matrices:

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 0 \\ 0 & 4 \\ 2 & -2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 0 & -6 \end{bmatrix}$$

If possible, find the following:

19. $A + B$

20. $B - A$

21. $A + C$

22. $5D$

23. CD

24. DC

25. BC

26. CB

27. $A + 2B$

28. $-AB$

In Exercises 29 and 30, find the inverse of the given matrix.

29. $\begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}$

30. $\begin{bmatrix} 1 & 1 & -4 \\ -5 & -2 & 0 \\ 4 & 2 & -1 \end{bmatrix}$

In Exercises 31 and 32, solve the given system by finding the inverse of the coefficient matrix.

31. $\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$

32. $\begin{aligned} x + 2y - 2z &= -4 \\ 3x - y \qquad &= -2 \\ y + 4z &= -1 \end{aligned}$

In Exercises 33–38, evaluate the determinant of the given matrix.

33. $\begin{vmatrix} 3 & 1 \\ -4 & 2 \end{vmatrix}$

34. $\begin{vmatrix} -1 & 2 \\ 0 & 6 \end{vmatrix}$

35. $\begin{vmatrix} 2 & -1 \\ 6 & -3 \end{vmatrix}$

36. $\begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & -5 \\ 0 & 4 & 0 \end{vmatrix}$

37. $\begin{vmatrix} 1 & -1 & 2 \\ 0 & 5 & 4 \\ 2 & 3 & 8 \end{vmatrix}$

38. $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{vmatrix}$

In Exercises 39–44, use Cramer's Rule to solve the given linear system.

39. $\begin{aligned} 2x - y &= -3 \\ -2x + 3y &= 11 \end{aligned}$

40. $\begin{aligned} 3x - y &= 7 \\ 2x + 5y &= -18 \end{aligned}$

41. $\begin{aligned} x + 2y &= 2 \\ 2x - 7y &= 48 \end{aligned}$

42. $\begin{aligned} 2x + 3y - z &= -3 \\ -3x \qquad + 4z &= 16 \\ 2y + 5z &= 9 \end{aligned}$

43. $\begin{aligned} 3x \qquad + z &= 0 \\ x + y + z &= 0 \\ -3y + 2z &= -4 \end{aligned}$

44. $\begin{aligned} 2x + 3y + z &= -5 \\ 2y + 2z &= -3 \\ 4x + y - 2z &= -2 \end{aligned}$

Review Test

Exercises 1 and 2 refer to the matrix

$$A = \begin{bmatrix} -1 & 2 \\ -2 & 4 \\ 0 & 7 \end{bmatrix}$$

1. Find the dimension of the matrix A.
2. Find a_{31}.
3. Write the augmented matrix of the linear system

$$\begin{aligned} -7x \qquad + 6z &= 3 \\ 2y - z &= 10 \\ x - y + z &= 5 \end{aligned}$$

4. Write a linear system corresponding to the augmented matrix

$$\left[\begin{array}{cc|c} -5 & 2 & 4 \\ 3 & -4 & 4 \end{array}\right]$$

5. Use back-substitution to solve the linear system corresponding to the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \end{array}\right]$$

6. Solve the linear system

$$\begin{aligned} -x + 2y &= 2 \\ \tfrac{1}{2}x + 2y &= -7 \end{aligned}$$

by applying Gaussian Elimination to the augmented matrix.

7. Solve the linear system

$$\begin{aligned} 2x - y + 3z &= 2 \\ x + 2y - z &= 1 \\ -x + y + 4z &= 2 \end{aligned}$$

by applying Gauss-Jordan Elimination to the augmented matrix.

8. Solve for x.
$$\begin{bmatrix} 2x-1 & 0 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & -3 \end{bmatrix}$$

Exercises 9–12 refer to the matrices

$$A = \begin{bmatrix} -4 & 0 & 3 \\ 6 & 2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

$$C = \begin{bmatrix} 4 & 2 \\ -2 & 0 \\ 3 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & -6 \\ 0 & 2 \\ 4 & -1 \end{bmatrix}$$

If possible, find the following:

9. $C - 2D$

10. AC

11. CB

12. BA

13. Find the inverse of the matrix
$$\begin{bmatrix} -1 & 0 & 4 \\ 2 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}$$

14. Solve the given linear system by finding the inverse of the coefficient matrix.
$$\begin{aligned} 3x - 2y &= -8 \\ 2x + 3y &= -1 \end{aligned}$$

In Exercises 15 and 16, evaluate the determinant of the given matrix.

15. $\begin{vmatrix} -6 & -2 \\ 2 & 1 \end{vmatrix}$

16. $\begin{vmatrix} 0 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & 4 & 5 \end{vmatrix}$

17. Use Cramer's Rule to solve the linear system
$$\begin{aligned} x + 2y &= -2 \\ -2x - 3y &= 1 \end{aligned}$$

Writing Exercises

1. Discuss how to solve a linear system in three unknowns if Cramer's Rule fails to hold.
2. Compare and contrast the additive properties of matrices with the additive properties of the real numbers.
3. Compare and contrast the multiplicative properties of square matrices with the multiplicative properties of the real numbers.
4. Compare and contrast Gauss-Jordan Elimination and Gaussian Elimination.

Chapter 8 Project

Manipulating images using computer technology is a major component of special effects in some of today's most popular films. The mathematics of matrices can help us see how images can be altered by increasing the contrast or adding two images together. One interesting use of the latter technique is a process called blue-screen chromakey, by which a character may appear to be in an environment which was actually photographed separately.

For this project, do the following exercises from this chapter: Section 8.1, 51, and Section 8.2, 37–39.

Now make up your own image matrix. Make it 20 pixels by 15 pixels, and let each pixel have 6 bits (this means each entry will be an integer between 0 and 63). Repeat the exercises using this matrix. Use your calculator to help you. If increasing the contrast results in an entry greater than 63, what should you do?